The Very Early Universe

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In these lectures [1] we dwell upon the cosmological corner-stones of the Very Early Universe (VEU) theory: Parametric Amplification Effect (PAE) responsible for the generation of Primordial Cosmological Perturbations (PCPs), Chaotic and Stochastic Inflation, Principal Tests of VEU, and others.

1 Introduction

A great success of VEU theory as the theory of the beginning of the Big Bang, is related to its semiclassical nature allowing to operate productively in terms of classical space-time filled with quantum physical fields (including the gravitational perturbations). It (VEU theory) connects like a bridge the theory of our Universe based on the Friedmann model (FU), with theories of Everything (TOE) essentially employing quantized gravity (still very ambiguous). This connection is already realized itself in the important understanding that the quasi-homogeneous and isotropic state of the Universe on the horizon scale (on one side) and the primordial cosmological perturbations which gave a birth to the Universe structure on smaller scales (on the other side) are just two features in the low-energy limit of some theory of VEU based on a model of the inflationary Universe (IU). Up to now we have no alternative to the Inflation resulting in the cold remnants which we observe today as the micro and macro worlds, which is commonly considered as a necessary element and the probe test of high energy physics and any TOE.

We should begin from the priorities when discussing VEU theory: whether to start from cosmological or particle physics standards. The particle physics is not yet fixed well at high energies: to follow this direction today means to start from N particle and modified gravity theories where N is a big number, and then to build up N inflationary models based on them. For this reason, rather preferable now seems the investigation of cosmologically standard VEU theories based of three standard points:

(i) setting some cosmological postulates within General Relativity (GR);

(ii) deriving the theory from these postulates, and

(iii) confronting the theory predictions with observations.

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On this way, which I will follow in these lectures, we are independent of the future particle physics and, thus, may try to find some basic properties and principal features of the VEU theory (tested by observations!) which stay independent of the particle physics uncertainties as well. We know two good examples of such kind theories. The first is the Friedmann-Robertson-Wolkel model which has ensured the great success of the observational and theoretical cosmology, just based on the cosmological postulate (the homogeneity and isotropy of the 3-space). The second is PAE, the theory of generation of PCPs which gave birth to the Large Scale Structure (LSS) of the Universe, just based on the linear perturbation theory in FU.

Regarding the VEU our goal would be to find the basics for IU in GR which could create the Friedmannian region we live in now (which, in its turn, would provide for the initial conditions for both our examples, the FU and PAE theories). The answer which we know today is the model of Chaotic Inflation based on the assumptions of existence of inflaton (the scalar field weakly coupled to all other physical field) and the start-inflation-condition postulating a quasi-homogeneous spatial distribution of the initial inflaton in a Compton-wavelength region (i.e., in the finite-scale region! A great step in comparison with the FU cosmological postulate !).

Below, our lectures will be grouped in four parts. The first is devoted to the PAE of scalar perturbations in FU (Lukash 1980). Next, we consider the cosmological applications to the *scattering* problem and chaotic inflation (Lukash & Novikov 1992). In the third part we present some recent developments of the theory of chaotic and stochastic inflation proposed by Linde (1983, 1986). The last Chapter deals with the problem of testing and confronting these theories with observations.

Certainly, we do not pretend to cover all the corresponding references and give a review of all recent ideas and speculations in VEU theories. Our main goal here is to present some basic properties of VEU which seem today more or less settled and independent of future theoretical constructions. We try to consider the simplest mathematic models paying particular attention to the physical meaning of the effects considered. Some necessary mathematical calculations are given in three Appendices. Hereafter, our units imply $c = 8\pi G = \hbar = 1$ and $H_o = 100h\, km\, s^{-1} Mpc^{-1}$.

2 Parametric Amplification Effect

The formation of the structure of the Universe is one of the fundamental problems of the modern Cosmology. The two following properties of the present Universe are very important in understanding the physics of its early expansion.

(i) High degree of homogeneity and isotropy at large scales ($\delta\rho/\rho \lesssim 10^{-4}$ on the cosmological horizon) along with a well-developed structure on scales less than 0.01 of the horizon.

(ii) High specific entropy ($N_\gamma/N_b \simeq 10^9$) along with the mean baryon density being less than $\sim 10\%$ of the critic value ($\Omega_b \lesssim 0.1$).

Property (i) proves that the large scale structure of the Universe (galaxies, clusters and supercluster) stemmed from initially small amplitude perturbations of homogeneous and isotropic cosmic medium since it is the small perturbation that may grow up to the order of unity (and then form gravitationally bounded object) only when the horizon becomes many times larger than its linear scale. We do not know yet whether PCPs formed together with the cosmological Friedmann model at Planckian curvatures of whether they originated in the process of the

homogeneous and isotropic expansion which is described by the Classical General Relativity (CGR). In the first case we have no theory. However, an important point is that the quasi-Friedmannian Cauchy-hypersurface is already a classical object after Planckian time. So, if the PCPs are made evolutionary and their scales less than the Friedmann-hypersurface scale then they are likely to form at a semiclassical stage when the large-scale gravity was governed by the CGR equations. We know one example of this kind - it is inflation: galaxy-scale PCPs form at the very late stages of the inflationary expansion when the Friedmannian Cauchy-hypersurface (which forms the background of our local Universe today) has been already prepared by the inflation. Below, we will develop the theory of small potential (scalar) PCPs assuming the existence of Friedmann background model.

Also, it is quite evident that by no process could the Universe be born strictly homogeneous and isotropic: there always exist quantum fluctuations of metric and physical fields, the seed fluctuations could be of statistical, random character or they might be thermal, etc. An important point here is as follows: the inevitable minimum level of seed initial perturbations is always maintained by the quantum point-zero fluctuations of a quasi-homogeneous gravitating medium which bases the Friedmann spatial slice.

One of the basic implications of property (ii) is that in the past the Universe was hot and its expansion was governed by the gravitational field of intensively interacting relativistic particles. At this stage the matter represented to high degree of accuracy a hydrodynamic perfect fluid with equation of state $p = \epsilon/3$ (The Hot Friedmann Universe, HFU).

Small perturbations of such radiationally dominated gravitating fluid are sound waves propagating through the matter with a constant amplitude (adiabatic decrease of the wave amplitude due to the Universe expansion is exactly compensated in this situation by the increase of the amplitude gained due to the pressure gradient in the comoving to the wave front reference system). Mathematically, such perturbations in the expanding Universe are governed by the same equation as that for the acoustic perturbations of a non-gravitating homogeneous static termal bath in the flat Minkowski space-time (conformal invariance). A deep physics is behind it: no new phonons, cosmological potential perturbations, can be produced during the expansion of the hot Universe.

For the real generation of PCPs to occur in the early Universe, one has to reject the hot equation of state ($p = \epsilon/3$) at some expansion epoch, which can be done in principle only in VEU before the primordial heating of cosmic matter to high relativistic temperatures. In this case, as we shall see below, the number of phonons is not conserved and new phonons (the density perturbations) can be created in the course of the expansion. The mechanism is purely classical and works as parametric amplification: energy of large-scale non-stationary gravitational potential is pumped to the energy of small-scale perturbations (like new photons are created in an electromagnetic resonator when its size changes in non-adiabatic way). This effect, which we will generally call "parametric amplification effect", has nothing to do with Jeans instability: actually it is the CGR-effect since the typical scales to be amplified are just the cosmological horizon (and the effect involves the light velocity and gravitational fundamental constants). Before we present the mathematical formalism the physical meaning of the parametric amplification is discussed in the next Section.

2.1 Physical Meaning of the Parametric Amplification

The theory of small perturbations in the FU was constructed by Lifshitz in 1946. According to this theory there are three types of perturbations of the homogeneous isotropic model: density

(potential) perturbations, vortex perturbations and gravitational waves. We are interested now in the first type of perturbations, potential perturbations, because they are related to galaxy formation.

Let us consider the spatially Euclidean background model:

$$ds^2 = dt^2 - a^2 d\vec{x}^2 = a^2 \left(d\eta^2 - d\vec{x}^2\right), \tag{1}$$

where a is the scale factor which is a function of time, t and η are universal and conformal times respectively,

$$\eta = \int \frac{dt}{a} \tag{2}$$

It is usual to present the density perturbations $\delta\rho$ in a Fourier expansion

$$\delta = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k}\delta_{\vec{k}} e^{i\vec{k}\vec{x}}, \tag{3}$$

where $\delta = \delta\rho/(\rho + p)$ (more rigorous definition which makes $\delta = \delta(t, \vec{x})$ the gauge invariant function is given later on). Here, the wavenumber $k = |\vec{k}|$ and the physical wavelength $\lambda = 2\pi a/k$.

Also we shall use the perturbation scale

$$l_k = \frac{\lambda}{2} = \frac{\pi a}{k}, \tag{4}$$

and the horizon scale[2]

$$l_H = H^{-1} = \frac{a}{\dot{a}} \tag{5}$$

The latter is a typical scale of the causally connected region during the evolution time scale of the a-function. (Dot is the derivative over the universal time $\dot{()} = \frac{d}{dt}$).

We consider here only homogeneous and isotropic states of the perturbation fields (random spatial phase fields). Their important characteristic is a power spectrum of the density perturbations Δ_k^2:

$$\frac{k^3}{2\pi^2}\langle\delta_{\vec{k}}\delta^*_{\vec{k}'}\rangle = \Delta_k^2\ \delta(\vec{k} - \vec{k}'), \tag{6}$$

where brackets $\langle ...\rangle$ mean average over the field state, $\delta(\vec{k} - \vec{k}')$ is the 3-dimensional δ-function and $(^*)$ is the complex conjugate. This spectrum determines the second correlation of the density perturbations

$$\xi(r) = \langle\delta(t, \vec{x})\delta(t, \vec{x} + \vec{r})\rangle = \int_o^\infty \frac{dk}{k}\Delta_k^2 \frac{\sin(kr)}{kr}, \tag{7}$$

where $r = |\vec{r}|$. All the odd correlations are identically zero. The Gaussian fields which are the particular cases of the random fields, can be totally described only by this correlation function (all higher order correlations are negligible in linear approximation). Eq. (7) also clarifies the physical sense of the power spectrum. The amplitudes Δ_k are just the corresponding density perturbations in the scale interval $\Delta k \sim k$ around k, they are additive (the total power is a sum of the partial ones over all scales):

$$\langle\delta^2\rangle = \sum_{\Delta k \sim k} \Delta_k^2$$

[2]For brevity, we refer to the Hubble scale H^{-1} as horizon, although it is not technically correct.

Let us now consider the hot Universe (HFU) when equation of state was [3] $p = \epsilon/3$. The solution for Δ_k is

$$\Delta_k^2 = (c_1(k) f_1(\kappa))^2 + (c_2(k) f_2(\kappa))^2, \tag{8}$$

where

$$\kappa = \omega\eta \sim \frac{l_H}{l_k} \sim \frac{t}{l_k} \sim a \sim t^{1/2}, \;\; \omega = \frac{k}{\sqrt{3}},$$

$$f_1(\kappa) = -\cos\kappa + 2\left(\frac{\sin\kappa}{\kappa} + \frac{\cos\kappa - 1}{\kappa^2}\right),$$

$$f_2(\kappa) = \sin\kappa + 2\left(\frac{\cos\kappa}{\kappa} - \frac{\sin\kappa}{\kappa^2}\right),$$

and $c_{1,2}(k)$ are the amplitudes of the growing and decaying modes respectively . The sound velocity here is $\beta = 1/\sqrt{3} \simeq 1$, so, κ function presents a ratio of the sound pass during the cosmological time t to the perturbation scale l_k which is about the horizon to perturbation scale ratio.

The perturbations of the gravitational potential or metric perturbations are as follows (exact definitions are given below):

$$h_1 = c_1 \frac{1 - \cos\kappa}{\kappa^2} \quad \text{for the growing mode,}$$

$$h_2 = c_2 \frac{\sin\kappa}{\kappa^2} \quad \text{for the decaying mode} \tag{9}$$

An important feature of these perturbations is the following. Neither growing nor decaying modes increase catastrophically in time. Both of them are described by sin and cos functions, so, if c_1 and c_2 are less than unity (and it should be this way, otherwise $h_{1,2}$ would be large at smaller time) then both these modes are just sound waves with constant in time amplitudes c_1 and c_2 and with different time phases.

First of all, this result means that the HFU is absolutely gravitationally stable against small perturbations: if initial perturbations are less than unity then they remain small forever till equation $p = \epsilon/3$ holds.

A more elegant proof of this important conclusion may be done with help of the q-scalar (Lukash, 1980) which is, generally, gauge invariant combination of matter velocity and gravitational perturbation potentials. Below, we shall see that potential perturbations in FU are totally described by this scalar (and back: all matter and metric perturbations can be expressed as functions of q). The physical meaning of the q-scalar easily follows from its definition: for large scale, $l_k > l_H$, q is mainly the gravitational potential (matter effects are not important), while inside horizon, $l_k < l_H$, gravitational perturbations are negligible and q is just the matter (velocity) potential.

In the HFU the q-field obeys the following equation:

$$\ddot{q} + 3H\dot{q} - \frac{1}{3a^2}\Delta q = 0, \tag{10}$$

where $H = \dot{a}/a$ is the Hubble function and $\Delta = \partial^2/\partial\vec{x}^2$ is the spatial Laplacian. Transformations $\bar{q} = aq$, $(') = d/d\eta = ad/dt$ reduce eq. (10) to (note, that $a \sim \eta$ for HFU)

$$\bar{q}'' - \frac{1}{3}\Delta\bar{q} = 0, \tag{11}$$

[3] In our units the energy density and matter density are the same functions, $\epsilon = \rho c^2 = \rho$.

which is just the non-gravitating acoustic wave equation in the flat spacetime (η, x):

$$\bar{q}_k \sim c_1 \sin\kappa + c_2 \cos\kappa. \tag{12}$$

Eqs. (1,10,11,12) indicate the conformal invariance of potential perturbations in HFU and, as a result, the conservation of the adiabatic invariant — the total number of phonons, the sound wave quanta — which proves the stability of the HFU expansion against small matter perturbations [4]. Note in this connection that gravitational waves have a similar invariance property (see Grichshuk 1974) but we do not discuss them here.

The lumps of the matter in these sound waves start growing only after the equality epoch ($l_H \sim 10^4 yrs$) when the non-relativistic particles become to dominate in the expansion and the pressure falls down in comparison with the total density. This process develops due to the Jeans gravitational instability causing the fragmentation of the medium into separate bodies at the late stages of the expansion ($l_H \sim 10^9 - 10^{10} yrs$).

We shall not discuss here these late processes of galaxy formation. For us the following is important: for the formation of large structures (superclusters and clusters), we need a definite amplitude of the sound waves $\sim 10^{-4} - 10^{-5}$ in the linear scales which encompass the number of baryons big enough for these structures formation. So, c_1 or/and c_2 must be of the order of 10^{-4} on these scales.

It is a very serious demand on the initial perturbations. Indeed, when t is small, $\kappa \ll 1$, we have (see eqs. (9,12)):

$$\begin{aligned} &c_1 \ll 1, \\ &c_2 \ll \kappa. \end{aligned} \tag{13}$$

From these expressions we can see that c_2 must be extremely small and cannot be of the order of 10^{-4}. So, we need in fact the following equations to be met for $\kappa \ll 1$:

$$\begin{aligned} &(i)\ c_1 \gg c_2, \\ &(ii)\ c_1 \sim 10^{-4}. \end{aligned} \tag{14}$$

But both of them look very strange.

Indeed, any general natural initial conditions assume a random time phase state for the seed fluctuations

$$c_1 = c_2 \ll\ 1. \tag{15}$$

E.g., the first eq. in (15) holds for vacuum or thermal fluctuations. More of that, any natural fluctuations in hot gravitating medium imply that c_1 and c_2 are dozers orders of magnitude less than 10^{-4}.

The last point is demonstrated with help of the following example. Let us suppose that the origin time of the fluctuations is the Planckian one and let us denote $k = 1$ for l_{pl}. Then on the galactic scale $k_{gal} \sim 10^{-26}$. Now, let us take a thermal fluctuation spectrum at this moment

[4] We do not go into further detail about this stability effect since we have emphasized it many times in our previous lectures. Mention only that the increase in time of the density contrast at $\kappa < 1$ (see eq. (8)) which some people interpret as an instability period, simply corresponds to the period of time of the monotonic change of the oscillatory function. (None of the sin-oscillations manage for $\kappa < 1$). Thus, to speak on the instability in this case is as incorrect as to speak on the instability of a mathematical pendulum when it moves, say, out of its stable point for the time which is less than the oscillatory period. Returning to our case, note that the potential energy of such a pendulum at $\kappa < 1$ is all in the gravity (see eqs. (9)). The Jeans thinking fails here because the $\kappa < 1$ region is purely relativistic one.

with the Planckian temperature and, thus, the maximum at $l \sim l_{pl}$. Then the amplitude of the perturbations for $k < 1$ would be proportional to $k^{3/2}$ and, on the galactic scale, it would be $\sim 10^{-40}$. So, c_1 has to be $\sim 10^{-40}$ and it is 35 orders of magnitude less than we need.

Our results are the following:

1) The classical cosmology of the hot VEU has principal difficulties in the explanation of origin of the PCPs. Both requirements provided by eqs. (14) for the large scale structure formation, cannot be naturally explained within the frameworks of the HFU.

2) To account for the appearance of the PCPs at the hot Universe expansion period we need, as a necessary condition, to reject the $p = \epsilon/3$ equation of state at the VEU stage. The modern cosmology provides for a variety of the possibilities of such type: from quantum-gravity effects to vacuum phase transitions, cosmic strings, textures, etc. Here we consider the most general conditions for the parametric amplification effect appearing in theories with one scalar field φ coupled to gravity in the minimal way.

Parametric amplification means the production of the gravitating potential inhomogeneities (PCPs) in a non-stationary gravitational background of the expanding Universe: large scale dynamic gravitational field parametrically creates (amplifies) the small scale perturbation fields. Mathematically, potential perturbations of FU with a general expansion law are governed by the q-scalar which, after the conformal transformations ($q \to \bar{q}$, $t \to \eta$), meets the following equation:

$$\Box_\beta \bar{q} = U\bar{q}, \tag{16}$$

where $\Box_\beta = \partial^2/\partial\eta^2 - \beta^2\Delta$ is the light ($\beta = 1$) or sound type ($\beta < 1$) d'Alambertian operator in the conformal spacetime and $U = U(\eta)$ is the effective potential of q-field, which is a function of the expansion rate of the FU. Eq. (16) is a type of the parametric equation in mathematical analysis capable to amplify the fields with scales $k \lesssim U^{1/2}$ which are usually outside or about the horizon size (i.e., in the purely relativistic region).

Say, for the massless scalar field φ with minimal coupling the effective potential is $U = a''/a$, thus, the typical frequency is just the horizon one (see eq. (16)):

$$\frac{U^{1/2}}{a} = \frac{\dot{(a^2H)}^{1/2}}{a} = H\left(2 - \frac{dl_H}{dt}\right)^{1/2} \sim H. \tag{17}$$

For the HFU, ($a \sim \eta$), the effective potential is identically zero $U = 0$, which reduces eq. (16) to eqs. (11,12) considered before. In this case we can define the vacuum state of the q-field for all spatial frequencies and introduce, for instance, a standard technics for the scattering problem with $|\,in\rangle$ and $|\,out\rangle$, vacua, and so on, to see how many phonons are spontaneously created during expansion, which are their spectrum, etc.

Further applications depend on the sign of the second derivative of the initial scale factor.

The point is that this sign can give us the idea about which scale expands faster: the perturbation scale l_k or the horizon l_H (see eqs. (4,5)). Indeed, the first derivative of their ratio is just proportional to the second derivative of the scale factor:

$$\dot{\left(\frac{l_k}{l_H}\right)} \sim \ddot{a}. \tag{18}$$

So, if $\ddot{a} < 0$ at the beginning, then the galactic scales are found initially outside the horizon, and the Cauchy initial data should be set up outside the horizon as well. On the contrary, if $\ddot{a} > 0$, then the initial conditions for the scales of interest can be set up inside the horizon.

We shall investigate both cases. The qualitative result is as follows: under natural initial conditions met by eq. (15) (e.g., $\mid in\rangle$ vacuum state for the q-field) it is the growing mode of perturbations that is finally created in the $\mid out\rangle$ state due to the parametric effect. So, the resulting perturbation field is described by the first line of eq.(14) with the $c_1(k)$ spectrum depending on the expansion factor behaviour at time period when the parametric amplification condition was met ($k \leq U^{1/2} \sim aH$).

The Lagrangian theory and the quantization of potential perturbations in FU are considered below. The next chapter deals with some cosmological applications.

2.2 Lagrangian Theory and Perturbations

Let us consider a scalar field $\varphi = \varphi(x^i)$ with the Lagrangian density depending on φ and its first derivatives in the following general form:

$$L = L(w, \varphi), \quad w^2 = \varphi_{,i}\varphi^{,i} = \varphi_{,i}\varphi_{,k}g^{ik}, \tag{19}$$

The action of the gravitating φ field is as follows:

$$W[\varphi, g^{ik}] = \int (L - \frac{1}{2}R)\sqrt{-g}\ \ d^4x, \tag{20}$$

where g_{ik} and R_{ik} are the metric and Ricci tensors respectively, $R = R^i_i$, $g = det(g_{ik})$. Variations of eq. (20) over φ and g^{ik} in extremum give the clasical equations of motions of the φ field

$$\left(\frac{n}{w}\varphi^{,i}\right)_{;i} + n\nu = 0, \tag{21}$$

and of the gravitational field created by the φ-field-source

$$G_{ik} = T_{ik}, \quad T_{ik} = \frac{n}{w}\varphi_{,i}\varphi_{,k} - g_{ik}L, \tag{22}$$

where $n = \partial L/\partial w$, $n\nu = -\partial L/\partial\varphi$, $G_{ik} = R_{ik} - Rg_{ik}/2$ and (;) is the covariant derivative in metric g_{ik}. Note, that eq. (21) can be obtained from the Bianchi identities $T^k_{i;k} = 0$, as well.

Useful constructions are the comoving (to φ-field) energy density and the total pressure of the φ field

$$\begin{aligned} \epsilon &= T_{ik}\varphi^{,i}\varphi^{,k}/w^2 = nw - L, \\ p &= \frac{1}{3}(\epsilon - T) = L, \end{aligned} \tag{23}$$

where $T = T^i_i$. Also, the following equations are valued

$$\begin{aligned} \epsilon + p = nw, \quad \beta^{-2} = \frac{w}{n}\frac{\partial^2 L}{\partial w^2} = \frac{\partial \ln n}{\partial \ln w}, \\ m^2 = -\frac{w}{n}\frac{\partial^2 L}{\partial \varphi^2}, \quad \Gamma = \frac{w}{n}\frac{\partial^2 L}{\partial w \partial\varphi} = w\frac{\partial \ln n}{\partial \varphi}. \end{aligned} \tag{24}$$

(Functions m^2 and β^2 can be negative).

When considering linear perturbation theory φ and g^{ik} are presented as sums of some known functions (the background (^{o}) solution) and small perturbations ϕ and h_{ik}:

$$\varphi = \varphi^{(o)} + \phi, \quad g^{ik} = g^{ik(o)} - h^{ik}. \tag{25}$$

Below, we consider the classical backgrounds (eqs. (21,22) are met automatically in (^{o}) order) and the perturbations can be quantum ones.

The Lagrangian of perturbation field is got by expanding the integrand of eq. (20) up to the second order in ϕ and h^{ik} with the total divergent terms excluded (see Appendix A):

$$W^{(2)}[\phi, h_i^k] = \int L^{(2)} \sqrt{-g^o}\, d^4x,$$

$$L^{(2)} = L^{(2)}(v, \psi_i^k) = \frac{\epsilon + p}{2}[v_i v^i + \chi^2(\beta^{-2} - 1) - 2v_i \psi_k^i u^k + v(\nu\psi - m^2 v + 2\Gamma\chi)] +$$
$$+ \frac{\epsilon - p}{2}(\psi_{ik}\psi^{ik} - \frac{1}{2}\psi^2) + \frac{1}{8}(\psi_{ik;l}\psi^{ik;l} - 2\psi_{ik;l}\psi^{il;k} - \frac{1}{2}\psi_{,l}\psi^{,l}), \tag{26}$$

where $v = \phi/w$, $v_i = v_{,i} + v(w_{,i}/w)$, $u_i = \varphi^{(o)}_{,i}/w^{(o)}$, $\chi = v_i u^i - h_{ik}u^i u^k/2 = \delta w/w$, $\psi_i^k = h_i^k - h\delta_i^k/2$, $h = h_i^i = -\psi$. (Hereafter all manipulations with indeces are carried out with help of the background metric tensors $g_{ik}^{(o)}$ and $g^{ik(o)}$, and background index (^{o}) is omitted where possible). Obviously,

$$\frac{\delta p}{\epsilon + p} = \chi - \nu v, \quad \frac{\delta \epsilon}{\epsilon + \rho} = \beta^{-2}\chi + (\nu + \Gamma)v. \tag{27}$$

The clasical field equations which couple the metric and scalar perturbations, can be obtained either when the first variations of the action (26) are taken equal to zero or, directly, while expanding eqs. (21,22) to the linear order terms. Generally, these equations describe three oscillators coupled to each other through the background shear and vorticity (6-order in time equation system): one oscillator is the scalar potential perturbations and the other two are just two polarizations of the gravitational waves. [5].

To find the physical degrees of freedom of the perturbation fields and to approach the problem of the PCP origin, the following steps have to be developed.

(i) Gauge invariant functions must be introduced instead of ϕ and h_{ik}

The point is that, although the original fields, scalar φ and tensor g_{ik}, are genuine by definition, their decomposition into background and perturbation parts is not unambiguous at all. Indeed, if we transform infinitesimally the reference system,

$$x^i = \tilde{x}^i + \xi^i, \tag{28}$$

where $\xi^i = \xi^i(x^k)$ are small arbitrary functions, then the new separation in the coordinates $\tilde{x}^i$ will take the following form:

$$\varphi = \varphi^{(o)}(x^i) + \phi = \varphi^{(o)}(\tilde{x}^i) + \tilde{\phi}, \quad \tilde{\phi} = \phi + w\xi_i u^i,$$
$$g_{ik}dx^i dx^k = \tilde{g}_{ik}d\tilde{x}^i d\tilde{x}^k, \quad \tilde{h}_{ik} = h_{ik} + \xi_{i;k} + \xi_{k;i}, \tag{29}$$

where the background metric $g_{ik}^{(o)}$ has the same functional dependence in the new coordinates $g_{ik}^{(o)}(\tilde{x}^l)$ as that in the old ones $g_{ik}^{(o)}(x^l)$.

To develop the gauge invariant theory one has, first, to expend the perturbation tensor h_{ik} over the irreducible representations of the background geometry to mark off the scalar and gravitational wave polarizations and, second, to find the appropriate gauge invariant (i.e., independent of the transformations (29)) linear superpositions of the perturbation functions.

[5]The vortex perturbations (if any) are standardly found as the first integrals of the 6-order equation system

(ii) The Lagrangian and Hamiltonian formalisms of the perturbation fields should be developed on the basis of the gauge invariant theory.

An important point here is to obtain the canonical field variables accounting for the physical degrees of freedom.

(iii) Secondary quantization of the perturbations and cosmological applications can be considered in connection of the PCP problem.

Here, we are going to analyze all these points for FU background metrics. In this case

$$
\begin{gathered}
g_{oi} = u_i = \delta_i^o, \quad g_{\alpha\beta} = -a^2\gamma_{\alpha\beta}, \\
H^2 = \frac{1}{3}\epsilon - \frac{K}{a^2}, \quad \dot{H} = -\frac{\epsilon+p}{2} + \frac{K}{a^2}, \\
\frac{\dot{n}}{n} + 3H + \nu = 0, \quad v_i = v_{,i} - \beta^2 v(3H+\nu)u_i, \\
\chi = \dot{v} - \beta^2 v(3H+\nu+\Gamma), \quad \frac{\delta\epsilon}{\epsilon+p} = \beta^{-2}\dot{v} - 3Hv,
\end{gathered}
\tag{30}
$$

where $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(x^\gamma)$ is the metric tensor of the homogeneous isotropic 3-space with the spatial curvature $K = 0, \pm 1$ (manipulations with the Greek indices are done with help of $\gamma_{\alpha\beta}$). All the perturbation types evolve independently of each other in the linear approximation since the background shear and vorticity are identically zero. Below, only potential perturbations are considered.

To avoid formal mathematical constructions we try to use here spatially flat FU ($\gamma_{\alpha\beta} = \delta_{\alpha\beta},\ K = 0$, see eq. (1)) and synchronous reference system for its perturbations ($h_{oi} = 0$) if no other cases are pointed out explicitly. The general necessary formulae are given in Appendix B.

2.3 Potential Perturbations in Friedmann Cosmology

Let us obtain equations of motion of the gauge invariant potential perturbations directly from the linearized Einstein eqs. (22), and their Lagrangian from eqs. (26).

General metric perturbations in the synchronous reference system are presented in terms of two gravitational potential, A and B:

$$
\begin{gathered}
ds^2 = dt^2 - a^2(\delta_{\alpha\beta} + h_{\alpha\beta})dx^\alpha dx^\beta, \\
h_{\alpha\beta} = A\delta_{\alpha\beta} + B_{,\alpha\beta}.
\end{gathered}
\tag{31}
$$

Since this metric is governed by the scalar field (19), we have three perturbation potentials v, A and B entering the field equations. In fact, only one of them is independent.

We will consider as the independent one a gauge invariant scalar q which is a linear combination of the perturbation potentials. Let us first define the q-scalar and then, using the low order equations, relate inversely this scalar to v, A and B. The gauge freedom in the choice of these potentials follows from eqs. (29):

$$
\tilde{v} = v + \frac{F}{2}, \quad \tilde{A} = A + HF, \quad \tilde{B} = B + F\int\frac{dt}{a^2} + G,
\tag{32}
$$

where F and G are small arbitrary functions of the space coordinates, and

$$2\xi_i = Fu_i - a^2(F_{,k}\int\frac{dt}{a^2} + G_{,k})P_i^k, \tag{33}$$

is the most general form of the ξ_i-vector in the synchronous gauge (Landau & Lifshitz 1967), $P_i^k = \delta_i^k - u_i u^k$ is the projection tensor.

Eqs. (32) show that the following function is gauge invariant:

$$q = A - 2Hv, \tag{34}$$

In fact, this function is independent of any other gauge as well (see Appendix B) which proves that $q = q(x^i)$ is a 4-scalar in the unperturbed FU.

To derive the inverse transformations and the equation of motion for q-field we will need only the low-order (in time) Einstein equations:

$$\delta G_o^o = H\dot{h} - \frac{\Delta A}{a^2} = \delta\epsilon, \tag{35a}$$

$$G_{o\alpha} = -\dot{A}_{,\alpha} = (\epsilon + p)v_{,\alpha}, \tag{35b}$$

$$\delta G_\alpha^\beta - \frac{1}{3}\delta_\alpha^\beta\delta G_\gamma^\gamma = \frac{1}{2a^2}(C_{,\alpha}^{,\beta} - \frac{1}{3}\Delta C\delta_\alpha^\beta) = 0, \tag{35c}$$

where $h = 3A + \Delta B$ and $C = A - (\dot{B}a^3)^{\dot{}}/a$. The first two eqs. (35a,b) are just the conservations of energy and momentum, respectively, whereas the last one (35c) states the Pascalian condition (the absence of pressure anisotropies for φ-field). In the class of functions under interest eq. (35c) have only the trivial solution[6]:

$$C = 0 \tag{36}$$

which relates A and B potentials straightforwardly.

Making use of eq. (35b), we can now express functions v, A and B in terms of the q-scalar:

$$v = \frac{1}{2}(Q - \frac{q}{H}), \quad A = HQ, \quad \dot{B} = a^{-2}Q - a^{-3}P,$$

$$Q = \int\gamma q dt, \quad P = \int a\gamma q dt,$$

$$\gamma = -\frac{\dot{H}}{H^2} = \frac{\epsilon + p}{2H^2} = \frac{3}{2}\frac{\epsilon + p}{\epsilon}. \tag{37}$$

Functions $Q = Q(x^i)$ and $P = P(x^i)$ are the q-integrals over the Friedmannian world line $dt = u_i dx^i$. They are determined up to the accuracy of some additive functions of the space coordinates. This freedom for the Q-function is just the gauge one (see (32)). The P scalar is gauge invariant, so its "ambiguity" is physically meaningfull and related to a certain class of perturbations of the Bianchi type I model. To prove it we need another relation between q and P functions which we are going to get from eq. (35a).

Let us recover the energy perturbation using eqs. (30, 37):

$$\frac{\delta\epsilon}{\epsilon + p} = -\frac{\dot{q}}{2H\beta^2} - 3Hv. \tag{38}$$

[6]The formal solution is $C = f\vec{x}^2 + \vec{g}\vec{x} + e$, where $f, \vec{g}$ and e are functions of time. The last two terms in the r.h.s. can be excluded because they do not enter the original $h_{\alpha\beta}$ functions. The quadratic term can be ascribed only to B potential which results in $h_{\alpha\beta} \sim \tilde{f}(t)\delta_{\alpha\beta}$, the latter excluded by redefining the scale factor.

Now, substituting it into eq.(35a), we have the key equation for q-scalar:

$$\gamma a^3 \dot{q} = \beta^2 \Delta P \tag{39}$$

which is obviously the GR-analog of Poison equation.

Let us first assume that $\beta \neq 0$. Comparison of eqs. (37, 39) shows that $P(t, \vec{x})$ is specified by the q-scalar up to accuracy of additive harmonic function of spatial coordinates $P(\vec{x})$:

$$\Delta P(\vec{x}) = 0. \tag{40}$$

In the class of the uniformly limited (in 3-space) functions $h_{\alpha\beta} \sim P(\vec{x})_{,\alpha\beta}$ the solution is a bilinear form with zero trace:

$$P(\vec{x}) = a_{\alpha\beta} x^\alpha x^\beta, \quad a_{\alpha\beta} = \text{const}, \quad a^\alpha_\alpha = 0. \tag{41}$$

Thus, potentials v, A and B are reconstructed from the given q-scalar but a partial solution that does not vanish under gauge transformations:

$$v = A = 0, \quad B = P(\vec{x}) \int \frac{dt}{a^3}. \tag{42}$$

Appendix C demonstrates that these perturbations (42) are homogeneous and belong to the Bianchi type I cosmological model.

For $\beta = 0$, function $P(\vec{x})$ is arbitrary and eqs. (42) describe the decaying mode of perturbations. The growing mode is determined by another arbitrary function of the space coordinates, $q(\vec{x})$. The general solution in this case is

$$v = 0, \quad Q = \frac{q(\vec{x})}{H},$$

$$P = q(\vec{x}) \int a\gamma dt + P(\vec{x}), \quad \delta\epsilon = \frac{H \Delta P}{a^3}. \tag{43}$$

So, with all the above said we may conclude that, for $\beta \neq 0$, the q-scalar is totally responsible for the evolution of the physical potential perturbations in spatially flat Friedmann model. Division over β^2 and differention of eq. (39) (with eq. (37) for the P-scalar taking into account) gives the second-order equation for the q-scalar:

$$\ddot{q} + \left(3H + 2\frac{\dot{\alpha}}{\alpha}\right)\dot{q} - \left(\frac{\beta}{a}\right)^2 \Delta q = 0, \tag{44}$$

where $\alpha^2 = \gamma/2\beta^2$.

Now, we can derive the Lagrangian density for the q-field. Substituting eqs. (37) into eq. (26) and leaving out full divergent terms, we have after rather lengthy calculations:

$$W^{(2)}[\phi, h^k_i] = W[q] = \int L(q) a^3 dt d^3x,$$

$$L(q) = \frac{1}{2}\alpha^2 \left(\dot{q}^2 - \left(\frac{\beta}{a}\right)^2 q_{,\alpha} q^{,\alpha}\right), \tag{45}$$

where $L(q)$ and $L^{(2)}$ differ each from the other only in the divergent terms.

Eqs. (45) evidence that q-scalar is the unique single canonical variable for physical degree of freedom of potential perturbations in the FU driven by scalar field of type (19). The Lagrangian density depends only on the first derivatives of q-field. However, if $\alpha \neq$ const then q acquires a mass. Endeed, introducing the following transformation

$$\tilde{q} = \alpha q, \tag{46}$$

we may rewrite the Lagrangian in the form of standard scalar field with square-mass $\mu^2 = -\ddot{\alpha}/\alpha$:

$$L(q) = \frac{1}{2}\left(\dot{\tilde{q}} - \left(\frac{\beta}{a}\right)^2 \tilde{q}_{,\alpha}\tilde{q}^{,\alpha} - \mu^2\tilde{q}^2\right) \tag{47}$$

Appendix B confirms eqs. (44,45) for general case. The covariant generalization is as follows:

$$(D^{ik}q_{,i})_{;k} = 0 \quad D_{ik} = \frac{1}{2}\alpha^2(u_iu_k + \beta^2 P_{ik}), \tag{48}$$

$$L(q) = \frac{1}{2}D^{ik}q_{,i}q_{,k}, \quad W[q] = \int L(q)\sqrt{-g}\; d^4x, \tag{49}$$

where $\alpha = (\epsilon + p)^{1/2}/2\beta H$, $P_{ik} = g_{ik} - u_iu_k$, $g = \det(g_{ik})$, all Friedmann functions.

Before going to the next point, we give some relations for density perturbations in different systems most frequently used in literature.

For the synchronous gauge we have from eq. (38):

$$\frac{\delta\epsilon}{\epsilon + p} = -\frac{\dot{q}}{2H\beta^2} + \frac{3}{2}(q - HQ). \tag{50}$$

For the comoving reference system with the synchronized time ($\tilde{v}_i \sim u_i$):

$$\frac{\delta\tilde{\epsilon}}{\epsilon + p} = -\frac{\dot{q}}{2H\beta^2}, \tag{51}$$

where

$$\begin{gathered} d\tilde{s}^2 = \left(1 + \frac{\dot{q}}{H}\right) d\tilde{t}^2 - a^2(\delta_{\alpha\beta}(1 + q) + \tilde{B}_{,\alpha\beta})d\tilde{x}^\alpha d\tilde{x}^\beta, \\ \tilde{t} = t + v, \quad \tilde{x}^\alpha = x^\alpha + \int \frac{v^{,\alpha}}{a^2}dt, \\ \dot{\tilde{B}} = \frac{q}{Ha^2} - \frac{P}{a^3}. \end{gathered} \tag{52}$$

For the Newtonian gauge (zero shear reference system, $\tilde{\tilde{B}} = 0$):

$$\frac{\delta\tilde{\tilde{\epsilon}}}{\epsilon + p} = -\frac{\dot{q}}{2H\beta^2} + \frac{3}{2}(q - \frac{HP}{a}), \tag{53}$$

where

$$\begin{gathered} d\tilde{\tilde{s}}^2 = \left(1 - \frac{HP}{a}\right) d\tilde{\tilde{t}}^2 - a^2\left(1 + \frac{HP}{a}\right)\delta_{\alpha\beta}d\tilde{\tilde{x}}^\alpha d\tilde{\tilde{x}}^\beta \\ \tilde{\tilde{t}} = t + \frac{1}{2}a^2\dot{B}, \quad \tilde{\tilde{x}}^\alpha + \frac{1}{2}B^{,\alpha}. \end{gathered}$$

For scales in the horizon ($k\eta \gg 1$) all the three expressions for $\delta\epsilon$ coincide since the leading term is the first one ($\delta\epsilon \sim \dot{q}$). In the relativistic region ($k\eta \leq 1$), $\delta\epsilon$ depends explicitly on spatial slice given.

2.4 Quantization and Conformal Non-Invariance

Taking in mind eqs. (48,49) one can formally consider the q-field as a test scalar field in the Friedmann models. It allows for a standard development of the Hamiltonian formalism.

First, we can construct the Hilbert space of all complex solutions of eq. (48) with the scalar product

$$(q_1, q_2) = \int_{\Sigma} J^i_{12} d\Sigma_i,$$

$$J^i_{12} = iD^{ik}(q_1^* q_{2,k} - q^*_{1,k} q_2), \tag{54}$$

where $d\Sigma_i$ is the invariant measure on a Cauchy-hypersurface Σ. The integral in eq. (54) does not depend on Σ choice because of the 4-flux conservation law $J^i_{12;i} = 0$.

Next, the canonically conjugate gauge invariant scalar is introduced:

$$\sigma = \sigma(x^i) = \frac{\partial L(q)}{\partial \dot{q}} = \alpha^2 \dot{q}. \tag{55}$$

Further steps to the constructing the field Hamiltonian and canonical quantization are as simple as that in the case of any other scalar field. We would like to emphasize here two points.

The quantization is based on the simultaneous commutation relation for the canonically operators q and σ:

$$[q(t, \vec{x}), \sigma(t, \vec{x}^{'})] = q\sigma - \sigma q = i\sqrt{-g}\ \ \delta(\vec{x} - \vec{x}^{'}). \tag{56}$$

This equation can be compared with the commutator between the velocity potential and density perturbation operators of sound waves in the nongravitating static matter, $[v, \delta\epsilon] = i\delta(\vec{x} - \vec{x}^{'})$ (see Lifshitz & Pitaevski 1978).

It is worth while rewriting eq. (48) in terms of the conformal coordinates (η, x) for the conformal field $\bar{q}$:

$$\Box_\beta \bar{q} = U\bar{q}, \quad \bar{q} = \alpha a q, \tag{57}$$

where $\Box_\beta = \frac{\partial^2}{\partial\eta^2} - \beta^2\Delta$ is the d'Alambertian operator in the Minkowski metric $d\bar{s}^2 = ds^2/a^2 = d\eta^2 - d\vec{x}^2$. The function $U = U(\eta) = (\alpha a)^{''}/(\alpha a)$ plays a role of the effective potential of scalar perturbations in FU. It is an unambiguous function of the background expansion or, more precise, of the scale factor and its time derivatives up to the forth order. Note, that the Lagrangian for eq.(57)

$$\bar{L}(\bar{q}) = \frac{1}{2a^4}(\bar{q}^{'2} - \beta^2 \bar{q}_{,\alpha}\bar{q}^{,\alpha} + U\bar{q}^2) \tag{58}$$

coincides with $L(q)$ up to full divergent term.

The total energy of potential perturbations in the Friedmannian space $t = \mathrm{const}$ — the field Hamiltonian — can be also presented in terms of the conformal field:

$$\frac{\bar{H}}{a} = a^3 \int E d^3\vec{x},$$

$$E = \frac{1}{2a^4}(\bar{q}^{'2} + \beta^2 \bar{q}_{,\alpha}\bar{q}^{,\alpha} - U\bar{q}^2), \tag{59}$$

where $E = E(\eta, \vec{x})$ is the local energy density of the q-field. Note, that for the non-gravitating matter (or for short wavelengths $k\eta \gg 1$) E is analogous to the sound wave energy density:

$$E \simeq \frac{(\epsilon + p)\vec{v}^2}{2} + \frac{(\beta\delta\epsilon)^2}{2(\epsilon + p)},$$

where $\vec{v} = (v^{,\alpha}/a)$ is the matter velocity.

Since this formal analogy with sound waves and the fact that PCPs, which are just the resulting (after amplification) q-field, are usually found at the beginning of the HFU expansion stage, we will call below the q-field quanta as phonons. These cosmological phonons remind the standard physics phonons only when phonon wavelength is inside the horizon ($k\eta \gg 1$) [7], in this case the gravity is negligent and $q \sim v$ (see eq. (34)). For large scales ($k\eta \leq 1$), matter effects are not important and q is mainly the gravitational field potential.

When scale factor is proportional to the conformal time, q-scalar appears conformally coupled to FU (see eq. (57)):

$$a \sim \eta/\alpha : \quad U(\eta) = 0. \tag{60}$$

In all the other cases $U \neq 0$ and the q-field is conformally non-invariant. It means that q interacts with background non-stationary metric, which provides for the spontaneous and induced production of phonons in the process of cosmological expansion.

The secondary quantization of q-scalar results in the following expansion:

$$q = \int d^3\vec{k}(a_{\vec{k}}q_{\vec{k}} + a^{\dagger}_{\vec{k}}q^*_{\vec{k}}), \tag{61}$$

where

$$(q_{\vec{k}}, q_{\vec{k}'}) = [a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = \delta(\vec{k} - \vec{k}'),$$

$$(q_{\vec{k}}, q^*_{\vec{k}'}) = [a_{\vec{k}}, a_{\vec{k}'}] = 0,$$

$a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$ are the annihilation and creation operators respectively,

$$q_{\vec{k}} = q_{\vec{k}}(x^i) = \frac{\nu_k}{(2\pi)^{3/2}\alpha a}\, e^{i\vec{k}\vec{x}},$$

and $\nu = \nu_k(\eta)$ satisfies the following equations

$$\nu_k^{''} + (\beta^2 k^2 - U)\nu_k = 0, \quad \nu_k \nu_k^{*'} - \nu_k^* v_k^{'} = i.$$

Below, we apply the theory of q-field for VEU. We shall assume $\mid in >$ vacuum initial state for the q-field. To define it explicity, we will consider in the next Chapter two cases for $\eta \to 0 :\ \ddot{a} < 0$ and $\ddot{a} > 0$.

3 Origin of Primordial Cosmological Perturbations

Here, we consider some cosmological applications of the theory of q-field: the scatterring problem and the problrm of the generation of PCPs in chaotic inflation.

Let us separate explicitely the kinetic and potential terms in the Lagrangian:

$$L = p(w) - V(\varphi). \tag{62}$$

From eqs. (23), we have

$$6a\ddot{a} = -(\epsilon + 3p) = 2V(\varphi) - (\epsilon(w) + 3p(w))\,.$$

[7] Rigorously speaking, inside the sound horizon, $k\eta \gg |\beta|^{-1}$ (see the d'Alambertian in eq. (57)). For estimates, we assume in the main text that $\beta \sim 1$.

where $\epsilon(w) = nw - p(w)$. It is seen, that if $V(\varphi) > 0$ and $\epsilon(\omega) > 0$, then the case $\ddot{a} < 0$ can be realized only when the potential term is negligible, whereas for the apposite case ($\ddot{a} > 0$) the potential term may play a dominant role. For this reason, we will consider two interesting for us asymptotics.

First, let us suppose that kinetic terms dominates the potential term in general Lagrangian. The following *theorem* can be easily proved in this connection:

The theory of a real scalar field with Lagrangian depending only on the kinetic term,

$$L = p(w), \quad w^2 = \varphi_{,i}\varphi^{,i}, \tag{63}$$

is mathematically equivalent to the theory of potential motions of the ideal fluid with arbitrary equation of state

$$p = p(w), \quad \epsilon = \epsilon(w) = w\frac{dp(w)}{dw} - p(w). \tag{64}$$

The 4-velocity of the ideal fluid is a time-like vector ($w > 0$):

$$u^i = \frac{dx^i}{ds} = \varphi^{,i}/w. \tag{65}$$

So, the φ-field acts here as the velocity potential.

In the other case $\ddot{a} > 0$, the potential term becomes important and we may decompose L-function over small parameter w^2:

$$L(w, \varphi) = -V(\varphi) + \frac{w^2}{2}W^2(\varphi) + 0(w^4). \tag{66a}$$

A simple redifinition of φ-field

$$\varphi \Rightarrow \int W(\varphi)d\varphi$$

reduces eq. (66a) to the following case with standard kinetic term :

$$L = \frac{1}{2}\varphi_{,i}\varphi^{,i} - V(\varphi). \tag{66b}$$

Next Section deals with a general scattering approach for the q-field (the first case can be solved only in this approximation). In the last Section of this Chapter we consider the Lagrangian (66b) with $V(\varphi) \gg |\varphi_{,i}\varphi^{,i}|$ initial condition.

3.1 Scattering Problem for q-Field

First, let us suppose that $V(\varphi) = 0$.

Applying the *theorem* to the flat FU, we see that for equation of state $p = \epsilon/3$ the q-field is conformally coupled ($U = 0$). Eqs. (59) give the integral of motion

$$a \sim \eta : \quad \bar{H} = \text{const}, \tag{67}$$

which means the conservation of the total number of phonons in the process of the cosmological expansion.

In fact, the phonon numbers conserve at any frequency mode. Let us dwell on it in a bit more detail.

At relativistic stage (67) phonons are presented by the following choice of functions [8]:

$$\nu_k = \frac{1}{\sqrt{2\omega}} \exp(-i\omega\tau), \quad \omega = \frac{k}{\sqrt{3}}, \tag{68}$$

where $a^{'} = \text{const}$, $a/a^{'} = \tau = \eta + \text{const}$, the prime ($'$) is the derivative in conformal time. The field Hamiltonian is a sum over all quanta energies:

$$H_{Reg} = \int d^3\vec{k} E_k N_{\vec{k}}, \tag{69}$$

where $E_k = \omega/a$ is the phonon energy and $N_{\vec{k}} = a^{\dagger}_{\vec{k}} a_{\vec{k}}$ is the operator of the number of phonons with physical momentum $\vec{k}/a$. The eigenvalues of the mean energy density operator

$$E = \frac{\bar{H}}{a^4 V}, \quad V = \int d^3\vec{x} \tag{70}$$

are as follows

$$E = \frac{1}{(2\pi a)^3} \int d^3\vec{k} E_k \left(n_{\vec{k}} + \frac{1}{2} \right), \tag{71}$$

where $n_{\vec{k}}$ are the occupation numbers of the phonon states.

Eq.(67) also allows for introduction of the growing and decaying mode operators:

$$C_1 \equiv C_1(\vec{k}) = \sqrt{\frac{\omega}{2}} \left(\frac{a_{\vec{k}} - a^{\dagger}_{-\vec{k}}}{ia'} \right),$$

$$C_2 \equiv C_2(\vec{k}) = \sqrt{\frac{\omega}{2}} \left(\frac{a_{\vec{k}} + a^{\dagger}_{-\vec{k}}}{a^{'}} \right). \tag{72}$$

In terms of these operators the field expansions have the following form (cf. eq. (12)):

$$q = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{k} e^{i\vec{k}\vec{x}} \left(C_1 \frac{\sin\omega\tau}{\omega\tau} + C_2 \frac{\cos\omega\tau}{\omega\tau} \right),$$

$$\bar{H} = \frac{1}{6}\epsilon a^4 \int d^3\vec{k} (\mid C_1 \mid^2 + \mid C_2 \mid^2), \tag{73}$$

where $\mid C \mid^2 = CC^{\dagger} = C^{\dagger}C$.

Now, let us calculate the number of phonons created at a period with some arbitrary expansion law governed by the general Lagrangian (19):

$$a = a(\eta), \quad \eta_1 < \eta < \eta_2. \tag{74}$$

We can do it directly taking into account the phonon numbers before ($\eta \leq \eta_1$) and after ($\eta \geq \eta_2$) this period and just comparing them (the scattering problem).

As we have already seen, it is possible to calculate the occupation numbers for any wavelength at a linear expansion stage (see eq. (67)). Thus, to solve our problem, we should match

[8] We do not go into detail about such standard for any quantized theory things as separation of the Hilbert space in the positive and negative frequency subspaces, the Fock space of states, Bogolubov transformations, etc.

the a-function (and its first derivative $\dot{a}$ [9]) by the linear passes $a \sim \eta$ at the beginning ($\eta = \eta_1$) and at the end ($\eta = \eta_2$) of the considered period.

So, in the resulting normalization (the Planck scale at Planck moment of time is $k^{-1} = 1$) we have:

$$a = \begin{cases} \tau = \eta = (2t)^{1/2}, & \eta \leq \eta_1 \\ a(\eta), & \eta_1 < \eta < \eta_2 \\ A_1\tau = A_1(\eta + \eta_o), & \eta > \eta_2, \ \eta_o = \text{const} \end{cases} \tag{75}$$

where $\tau = \tau(\eta)$ is the conformal horizon (Hubble) time:

$$\tau = \frac{a}{a^{'}} = \eta - \int_{\eta_1} \frac{aa^{''}}{a^{'2}} d\eta = \frac{1}{2} \int \left(1 + \frac{3p}{\epsilon}\right) d\eta. \tag{76}$$

The value of the physical constant

$$A_1 = \frac{a^{'}(\eta = \eta_2)}{a^{'}(\eta = \eta_1)} \tag{77}$$

depends on the average expansion rate at period (74). Locally, A_1-factor can be related to the conformal acceleration:

$$\frac{a^{''}}{a^{'}} = \lim_{\eta_2 \to \eta_1} \left(\frac{A_1 - 1}{\eta_2 - \eta_1}\right). \tag{78}$$

Let $a_{\vec{k}}$ and $b_{\vec{k}}$ be the phonon representations (68) diagalizing the Hamiltonian at stages $\eta \leq \eta_1$ and $\eta \geq \eta_2$ respectively. Then

$$b_{\vec{k}} = \alpha_k a_{\vec{k}} + \beta_k^* a_{-\vec{k}}^{\dagger}, \quad \mid \alpha_k \mid^2 - \mid \beta_k \mid^2 = 1, \tag{79}$$

where α_k and β_k are Bogolubov coefficients. The $\mid in\rangle$ and $\mid out\rangle$ vacua are defined accordingly:

$$a_k \mid in\rangle = 0, \quad b_k \mid out\rangle = 0. \tag{80}$$

The Heisenberg state of the q-field is supposed to coincide with the $\mid in\rangle$ vacuum. It means that there are no phonons (i.e., potential perturbations) initially.

Taking average over the $\mid in\rangle$ vacuum state, we derive the mean occupation numbers of phonons spontaneously created at period $\eta_1 < \eta < \eta_2$,

$$\langle b_{\vec{k}}^{\dagger} b_{\vec{k}} \rangle = n_k \delta(\vec{k} - \vec{k}^{'}), \quad n_k = \mid \beta_k \mid^2, \tag{81}$$

and ratio of the energy densities of the perturbations field to the homogeneous cosmological field (see eqs. (70,71)):

$$\frac{E_{Reg}}{\epsilon} = \frac{1}{24\pi^3 A_1^2} \int d^3\vec{k} \ \omega \mid \beta_k \mid^2. \tag{82}$$

The factor A_1^{-2} takes into account phonon energy cooling during the expansion period (74).

Calculations of the produced spectrum is also straightforward:

$$\langle q^2 \rangle = \int_o^{\infty} \frac{dk}{k} q_k^2$$

[9] We need this condition in order to avoid the unwanted non-physical effects of creation in the matching points. For further details about scattering problem, which are standard for any test field theory, see Grib et al. (1980), Birrel & Davies (1981), and others.

$$q_k^2 = \left(\frac{k}{\pi\tau A_1}\right)^2 (\mid \beta_k \mid^2 + Re(\alpha_k \beta_k^* e^{-2i\omega\tau})). \tag{83}$$

The spectrum dependence on the oscillatory exponent means that growing and decaying modes are created non-equally (see eq. (73)). Now, we are going to prove that under general condition $a^{''} > 0$ ($A_1 > 1$, see eq, (78)) which is most frequently met in the applications, it is the growing mode of perturbations that is preferably created by the parametric mechanism.

Indeed, our initial conditions generally impty ($\eta \leq \eta_1$, cf. eq.(15)):

$$C^2 = \langle \mid C_1 \mid^2 \rangle = \langle \mid C_2 \mid^2 \rangle \ll 1. \tag{84}$$

The situation is trivial in case of the acceleration ($\ddot{a} > 0$) when the perturbation scales inflate from inside to outside the horizon. The decaying mode, appearing originally at the horizon with the same amplitude as the growing one, decays quickly for larger times while the growing mode is frosen. It can be demonstrated with help of general solution of the dynamic eq. (46) in large scales (the Laplacian term is negligible):

$$q = q_1(\vec{x}) + q_2(\vec{x}) \int \frac{dt}{\alpha^2 a^3}, \tag{85}$$

where $q_{1,2}(\vec{x})$ are arbitrary functions of the spatial coordinates. The integral sharply converges to a constant in time function

$$q = q(\vec{x}), \tag{86}$$

which describes the growing perturbation mode at $\eta \mid \nabla q \mid \ll q$ for any expansion law.

Let us consider in a more detail the case when the initial conditions (84) are set up outside the horizon. The general solution (85) helps again. Comparing it with the a-representation functions $\nu_k = \nu_k(\eta)$ which have the form (68) for $\eta \leq \eta_1$, and

$$\nu_k = \frac{1}{\sqrt{2\omega}}(\alpha_k e^{-i\omega\tau} + \beta_k e^{i\omega\tau})$$

for $\eta \geq \eta_2$, we obtain for the Bogolubov coefficients at $k\tau_1 \ll 1$:

$$\alpha_k = \frac{1}{2}(A_1^{-1} + i g_k A_1), \quad \beta_k = \frac{1}{2}(A_1^{-1} - i g_k A_1), \tag{87}$$

where $g_k = \chi_1/k - i$ is the amplification coefficient,

$$\frac{\chi_1 \tau_1}{\sqrt{3}} = 1 - \frac{\tau_1}{\tau_2 A_1^2} - \int_{\eta_1}^{\eta_2} (\alpha a)^{-2} d\eta = \text{const}.$$

Obviously, $g_k = \chi_1/k$ for $g_k \gg 1$ and $g_k \sim (\omega\tau_1)^{-1}$ for $A_1 \gg 1$.

Substitution of eqs. (87) into eq. (83) gives the following spectrum for $A_1 \gg 1$:

$$q_k = \frac{k^2}{2\pi} \mid g_k \mid \frac{\sin \omega\tau}{\omega\tau}. \tag{88}$$

The comparison with eqs. (73) reveals easily that only the growing mode is created.

To clarify the physical meaning of this effect, let us relate directly the $C_{1,2}$ operators in a and b representations (see eqs. (72,79,87), $k\tau_1 \ll 1$):

$$C_1^{(b)} = C_1^{(a)} + \frac{\chi_1}{k} C_2^{(a)}, \quad C_2^{(b)} = A_1^{-2} C_2^{(a)}. \tag{89}$$

So, if one begins with eq. (84) then, in the end, eqs. (89) give for $A_1 \gg 1$ and $g_k \gg 1$

$$\langle | C_1 |^2 \rangle = g_k^2 C^2 \gg C^2 \gg \langle | C_2 |^2 \rangle, \tag{90}$$

what is just required by the galaxy formation theories (cf. eq.(14)). In fact, the effect is a pure game of the mode mixing, it is not the q-field itself that is created but rather the q-momentum (the time derivative $\dot{\bar{q}}$). In other words one can say that the parametric amplification effect brings about the creation of squeezed state from initially random state.

So, as we could see, it is not a problem to produce the growing mode of perturbations with necessary amplitude. The amplification coefficient is the larger the earlier HFU expansion is violated. The typical spectra (88) go like

$$q_k = Mk, \qquad k < M, \tag{91}$$

with the maximum amplitude corresponding to the mass scale $M \sim \tau_1^{-1}$ when the linear expansion law was broken for the first time. (Note, that the spectrum (91) decreases to large wavelengths in comparison with the Harrison-Zeldovich scale-free spectrum $q_{HZ} \sim$ const). For $k \gg M$, the amplification coefficient is exponentially small.

There are two points concerning eq. (91).

(i) Initial (vacuum) conditions are set up outside the horizon which requires physical explanation.

(ii) To get the expansion factor required for typical scales $k \leq M$ to be of the order of the galactic scales, it is necessary to ensure the acceleration (inflationary) condition $\ddot{a} > 0$ at period (74).

In the latter case, the initial vacuum condition must be set up in the adiabatic zone — within the horizon — which can be done independently of the expansion law at the beginning.

3.2 Generation of Perturbations on Inflation

Let us consider Lagrangian (66) with a potential $V = V(\varphi) \in C_3$ (at least, the first three derivatives determined). We shall generally assume it to be a monotonically growing function of φ for $\varphi > \varphi_o$. Without loss of generality we can put $\varphi_o = 0$ and $V = dV/d\varphi = 0$ at $\varphi = 0$ which, with the symmetric condition $V(\varphi) = V(-\varphi)$, makes a stable minimum of the potential $V(\varphi)$ at point $\varphi = 0$. Under such normalization, our main assumption takes the following form:

$$V > 0, \qquad \frac{dV}{d\varphi} > 0 \ \text{ for } \ \varphi > 0. \tag{92}$$

Also, we define three auxiliary functions of φ related to the potential derivatives:

$$c = \frac{d\ln V}{d\ln\varphi}, \quad e = \frac{d\ln c}{d\ln\varphi}, \quad f = \frac{d\ln e}{d\ln\varphi}.$$

The simplest examples are $V_1 = m^2\varphi^2/2$ and $V_2 = \lambda\varphi^4/4$ where constants m, λ are the field mass and dimensionless parameter, respectively. Evidently, for the power-law potentials $V_n \sim \varphi^{2n}$, $c = 2n =$ const, and $e = f = 0$.

The background Friedmann quantities and eqs. (30) are

$$\frac{2\alpha^2}{3} = \left(1 + \frac{2V}{\dot{\varphi}^2}\right), \tag{93}$$

$$\beta^2 = 1, \quad w = n = -\dot{\varphi}, \quad p = -V + \frac{\dot{\varphi}^2}{2}, \tag{94}$$

$$\epsilon = 3H^2 = V + \frac{\dot{\varphi}^2}{2}, \quad \dot{H} = -\frac{\dot{\varphi}^2}{2}, \tag{95}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0. \tag{96}$$

Inflation occurs when

$$\ddot{a} = a(\dot{H} + H^2) = \frac{a}{3}(V - \dot{\varphi}^2) > 0. \tag{97}$$

At $V > \dot{\varphi}^2$ the potential energy density contributes dominantly to the Hubble parameter. For our functions (92) it may happen at large φ, and inflationary solution can be got as expansion over the inverse powers of $\mid \varphi \mid \gg 1$:

$$H = \sqrt{\frac{V}{3}}(1 + \frac{1}{12}c^2\varphi^{-2} + O(\varphi^{-4})), \tag{98}$$

$$\dot{\varphi} = -\frac{cH}{\varphi}(1 + \frac{1}{3}(e-1)c\varphi^{-2} + O(\varphi^{-4})), \tag{99}$$

$$a = \exp\left[-\int d\varphi \;\; \varphi(c^{-1} + \frac{1}{3}(1-e)\varphi^{-2} + O(\varphi^{-4}))\right], \tag{100}$$

$$\alpha = \frac{c}{2\varphi}(1 + \frac{1}{3}(e-1)c\varphi^{-2} + O(\varphi^{-4})). \tag{101}$$

Then the inflationary condition can be rewritten in terms of c-function:

$$\left|\frac{c}{\varphi}\right| < \sqrt{3}, \tag{102}$$

or in terms of potential V:

$$\left|\frac{dV}{d\varphi}\right| < \sqrt{3}V. \tag{103}$$

Eqs. (98,99,100,101) are obviously true in a so-called slow-roll approximation which assumes friction-dominated equation of motion for φ-field (the second-derivative term in eq. (96) is subdominant). There are two conditions for this approximation following directly from expansions (98 – 101):

$$\left|\frac{c}{\varphi}\right| \le 2, \qquad \left|\frac{(e-1)c}{\varphi^2}\right| \le 2. \tag{104}$$

The first condition nearly coincides with eq. (102). From the second condition we have

$$\left|\frac{d^2V}{d\varphi^2}\right| \le 2V. \tag{105}$$

So, the inflationary solution (98, 99, 100, 101) requires certain analytical properties from potential $V(\varphi)$ (see eqs. (103,105)). We can put it in other terms: eqs. (98, 99, 100, 101) are valued for large φ,

$$|\varphi| \geq \varphi_I = \max\left(\frac{|c|}{2},\ \sqrt{\frac{|(e-1)c|}{2}}\right). \qquad ((106)$$

One can show that, within class of functions (92) satisfying the inequality (106) for

$$\mid \varphi \mid \geq \varphi_1 = \mathrm{const} \qquad (107)$$

where φ_1 is a positive root of equation $\varphi_1 = \varphi_I(\varphi_1)$ [10], eqs. (98, 99, 100, 101) describe a trap (for growing time) separatrix towards which all the other dynamical trajectories [11] with initial $\mid \varphi \mid > \varphi_1$ approach rapidly during their dynamical evolution to the stable point $\varphi = \dot{\varphi} = 0$. On the inflationary separatrix (98, 99, 100, 101) φ, H and α vary slowly while the scale factor increases nearly exponentially in time, $a \sim \exp(Ht)$.

Eqs. (98 – 101) break when the field reaches the point $\mid \varphi \mid \sim \varphi_1 \sim 1$ and the further evolution proceeds with damping oscillations around $\varphi = \dot{\varphi} = 0$. If $d^2V/d\varphi^2 > 0$ at $\varphi = 0$, then

$$H = \frac{2}{3}t^{-2}, \quad a \sim t^{2/3}, \quad \varphi = 2\sqrt{\frac{2}{3}}(mt)^{-1}\sin(m(t+t_o)) \leq 1, \qquad (108)$$

where $V \simeq m^2\varphi^2/2$ for $\mid \varphi \mid \leq 1$, here m and t_o are constants. At this stage the Universe expands like a pressureless medium since the average cosmological pressure is exponentially small. The medium — coherent oscillations of spatially homogeneous field — is unstable in this situation and will decay in particles. The result is reheating and the HFU expansion beginning.

This reheating process although producing some inhomogeneities on the horizon scale $\sim k_1$, cannot damage the large scale perturbations created already during the inflationary epoch ($k < k_1$).

To find the postinflationary PCP spectrum we must solve Eqs. (61) with parameters (98, 99, 100, 101) for $\varphi > \varphi_1$:

$$\nu_k'' + (k^2 - U)\nu_k = 0, \qquad (109)$$

$$U = \frac{(\alpha a)''}{\alpha a} = 2(aH)^2\left(1 + \frac{\dot{H}}{2H^2} + \frac{(a^3\dot{\alpha})^{\dot{}}}{2a^3\alpha H^2}\right) =$$

$$= 2(aH)^2\left(1 - \frac{1}{4}c_2\varphi^{-2} + O(\varphi^{-4})\right) =$$

$$= \frac{2}{\eta^2}\left(1 + \frac{3}{4}c_3\varphi^{-2} + O(\varphi^{-4})\right), \qquad (110)$$

$$\eta = \int\frac{dt}{a} = -\frac{1}{aH}\left(1 + aH\int\frac{\dot{H}dt}{aH^2}\right) = -\frac{1}{aH}\left(1 + \frac{1}{2}c^2\varphi^{-2} + O(\varphi^{-4})\right), \qquad (111)$$

[10] For estimates, $\varphi_1 \sim 1$. If there are few roots in eq. $\varphi = \varphi_I(\varphi)$ for $\varphi \geq 1$, then the solution (98, 99, 100, 101) can be broken for some large φ. We do not analyse here such possibility.

[11] In the phase space $(\varphi, \dot{\varphi})$, initial conditions for the classical trajectories of eqs. (95, 96) are set up on the quantum boundary $\epsilon = 1$ which represents a kind of ellipse (circle in case of V_1) around the central point $\varphi = \dot{\varphi} = 0$. The radii of this ellipse along φ and $\dot{\varphi}$ axes are V^{-1} [1] and $\sqrt{2}$, respectively.

where $c_2/c = c + 6(1-e)$, and $c_3/c = c + 2(e-1)$. The conformal time $\eta < 0$, and initial conditions are

$$\nu_k = \frac{1}{\sqrt{2k}} \exp(-ik\eta), \quad \text{for} \quad k \mid \eta \mid \gg 1. \tag{112}$$

Eqs. (109,110,111,112) can be solved explicitely by matching two following solutions in the overlapping region

$$c_4 \varphi^{-4} < k \mid \eta \mid < 1, \tag{113}$$

where $c_4/c^2 = (c + e - 2)(1 - e) - ef$. The first solution assuming the left inequality (113), allows for the U-potential approximation by $U = \text{const}/\eta^2$ near $k \mid \eta \mid \sim 1$ (cf. eq. (110)). Since $U \ll k^2$ for $k \mid \eta \mid \gg 1$, we have for $k \mid \eta \mid > c_4 \varphi^{-4}$:

$$\nu_k = \frac{1}{2}\sqrt{\pi \mid \eta \mid} H_\nu^{(1)}(k \mid \eta \mid) = \frac{1}{\sqrt{2k}}(1 - \frac{i}{k\eta}) \left(e^{-ik\eta} + O(\varphi^{-2}) \right), \tag{114}$$

where $H_\nu^{(1)}(x)$ is the Hankel function, $\nu = (3 + c_3 \varphi^{-2})/2$.

Under the right inequality (113), eq.(85) describes the general solution for any $U(\eta)$. Since the integral in eq. (85) converges sharply in time, we have for $k \mid \eta \mid < 1$:

$$\nu_k = i\pi\sqrt{2} k^{-3/2} \alpha a q_k = -\frac{i\pi}{\sqrt{2}\eta} k^{-3/2} \left(\frac{c}{\varphi H} \right) q_k, \tag{115}$$

where constants q_k are the PCP spectrum (see eq. (83)).

Fitting eqs. (114,115) at region (113) gives the following spectrum of the perturbations parametrically created outide the horizon:

$$q_k = \frac{1}{\pi} \left(\frac{\varphi H}{c} \right)_k = \frac{1}{\pi} \left(\frac{H^2}{\mid \dot{\varphi} \mid} \right)_k = \frac{1}{\pi\sqrt{3}} \left(\frac{V^{3/2}}{dV/d\varphi} \right)_k, \tag{116}$$

where $\varphi = \varphi_k$ at the horizon crossing ($k\eta = -1$) can be expressed directly in terms of the wave number

$$k = aH = a(\varphi)H(\varphi). \tag{117}$$

The resulting spectrum (116,117) belongs obviously to the growing perturbation mode since only for this mode q-field is constant in time outside the horizon. We have already emphasized that this property of the growing mode is independent of any expansion law or equation of matter state. In particular, spectrum (116,117) holds in large scales for any microphysics processes after inflation like phase transitions or reheating.

Sometimes, people prefer to deal with the power spectrum of density perturbations. Below, relation between q-scalar and the coupled density perturbation field is obtained in a general form.

Indeed, eq. (44) allows for the growing mode general solution outside horizon:

$$q = q(\vec{x}) + Q(t)\Delta q(\vec{x}),$$

$$\dot{Q}(t) = -\left(\frac{\beta}{a} \right)^2 \frac{H}{\dot{H}} (1 - \frac{H}{a} \int a dt). \tag{118}$$

So, the comoving density perturbations are (see eq. (52)):

$$\delta = \frac{\delta \tilde{\epsilon}}{e} = -\frac{2\alpha^2 \dot{q}}{3H} = \frac{1}{3} \frac{1}{(aH)^{-2}} (1 - \frac{H}{a} \int a dt) \Delta q(\vec{x}). \tag{119}$$

Now, it is not a problem to get the desired relation between power spectra:

$$\delta_k = \frac{q_k}{3}\left(\frac{k}{aH}\right)^2\left(1 - \frac{H}{a}\int adt\right) = q_k O\left(\left(\frac{l_H}{l_k}\right)^2\right) \ll q_k, \tag{120}$$

We return now to eq. (117) which gives us the connection between k and φ on the inflationary separatrix (98,99,100,101). This equation can be solved explicitly for a class of so-called smooth potentials $V(\varphi)$.

Let us introduce smooth potential functions $V(\varphi)$ for which $c(\varphi)$ varies even slower than φ [12]:

$$\mid e(\varphi) \mid \ll 1. \tag{121}$$

For such potential eq. (100) yields

$$a \simeq \varphi^{-1/3}\exp\left(-\frac{\varphi^2}{2c}\right), \quad \varphi \geq \varphi_1, \tag{122}$$

which shows that φ varies logarithmically in the conformal time (cf. eq. (111)). If $c_1 = c(\varphi_1) \geq 2$, then $\varphi_1 = c_1/2 \geq 1$ (see eqs. (106,107)) and

$$\varphi^2 = \frac{c}{2}\left(c_1 + 4\ln\left(\frac{\eta H \varphi_1^{1/3}}{\eta_1 H_1 \varphi^{1/3}}\right)\right), \tag{123}$$

for $\varphi \geq \varphi_1$ ($\mid \eta \mid \geq \mid \eta_1 \mid$). The substitution to eq. (117) gives for $k \leq k_1$:

$$\varphi_k^2 = \frac{c}{2}\left(c_1 + 4\ln\left(\frac{k_1 H \varphi_1^{1/3}}{k H_1 \varphi^{1/3}}\right)\right) \simeq 2c\ln(k_1/k), \tag{124}$$

(the second equality implies $k \ll k_1$).

So, smooth potentials generate the Harrison-Zeldovich types of spectra (see eq. (116)) growing only logarithmically to large scales.

Typical example of smooth potential is a power-law potential

$$V_n = \frac{1}{2n}a_n^2\varphi^{2n}$$

which generates the following spectrum:

$$q_k = (4\pi n^{3/2})^{-1}a_n\varphi^{n+1} =$$

$$= \frac{2}{\pi}(4n)^{(n-2)/2}a_n\left[n/2 + \ln\left(\frac{k_1}{k}\left(\ln\frac{k_1}{k}\right)^{(3n-1)/6}\right)\right]^{(n+1)/2}, \tag{125}$$

where a_n and $c = c_1 = 2n \geq 2$ are constants.

Important cases are the massive field ($n = 1,\ a_1 = m$):

$$q_k = \frac{m}{\pi}\ln\left(\frac{k_1}{k}\left(\ln\frac{k_1}{k}\right)^{1/3}\right), \tag{126}$$

[12] Physically, the characteristic scales of smooth potentials are not much shorter than φ.

and the λ-field ($n = 2,\ a_2 = \sqrt{\lambda}$):

$$q_k = \frac{2\sqrt{\lambda}}{\pi}\left[\ln\left(\frac{k_1}{k}\left(\ln\frac{k_1}{k}\right)^{5/6}\right)\right]^{3/2}. \tag{127}$$

For non-smooth $V(\varphi)$ spectrum q_k can have, in principle, any form depending on given potential and the first derivative shapes. Moreover, it is possible to inverse the problem and to find potential $V(\varphi)$ for any [13] given postinflationary PCP spectrum (Hodges & Blumenthal 1989). True, some potentials appear to be rather exotic ones, but the result is very important: PCP spectra are very sensitive to the potential forms [14].

We shall return to the latter problem in the fifth Chapter. But now, let us stress two more points in the conclusion.

Postinflationary perturbations (116,117) are Gaussian with random spatial phases since it is the seed point-zero vacuum fluctuations (of the q-field) from which they were parametrically created, that are Gaussian by definition. Here, we have no problem with initial conditions for the q-scalar because they are determined by microphysics inside the horizon.

Another interesting point is that most spectra grow with scale growing. It means that there exists some critical field (and, thus, the critical scale) for which the corresponding amplitude $q_k \sim 1$:

$$\left(\frac{\varphi H}{c}\right)_{cr} \sim 1, \quad k_{cr} \sim (aH)_{cr} \ll k_1. \tag{128}$$

Say, for potentials (125,126,127) we have

$n \geq 1:\quad (\varphi H/c)_{cr} \sim 1, \qquad k_1/k_{cr} \sim \exp(a_n^{-2/(n+1)})$,
$n = 1:\quad \varphi_{cr} \simeq 2\sqrt{\pi}m^{-1/2}, \quad k_1/k_{cr} \simeq (m/\pi)^{1/3}\exp(\pi/m)$,
$n = 2:\quad \varphi_{cr} \simeq 3\lambda^{-1/6}, \qquad k_1/k_{cr} \sim \exp(\lambda^{-1/3})$.

We shall see in the next Chapter that the Universe on large scales, $k \leq k_{cr.}$, is globally non-linear and it is stochastic (dominated by quantum fluctuations) for $\varphi > \varphi_{cr.}$.

4 Inflation

There is no secret that Inflation is a corner stone of the VEU theories. This is not a surprise since up to now we have no alternative to the Inflationary Paradigm.

We are not going to discuss the Paradigm here. There is a lot of reviews and courses devoted to the subject (e.g. see Colb & Turner 1989, and references therein). Instead, we would like to dwell on the chaotic inflation which, in our view, is the first theory of the kind that can be called the cosmologically standard theory. At least, in a sense as this status has the standard Friedmann cosmology or the parametric amplification theory. All of them, based on simple cosmological postulates which are not directly related to any particular particle physics, can explain and predict a lot of obvservational consequences (see the Introduction).

The goal of inflationary theory is to prepare initial conditions for the standard FU. There are the following five items among them.

(i) Homogeneity and isotropy along with the Euclidean geometry of the spatial slice on scales near the contemporary horizon.

[13] Some spectra may violate the slow-roll conditions which makes the inverse problem self-inconsistent.

[14] Physically, non-Harrison-Zeldovich spectra appear from potentials which have characteristic scales less then φ.

(ii) The amplitude $\sim 10^{-4}$ of PCPs at this slice on galactic to supercluster scales.

(iii) Reheating sufficient for the primordial entropy production, nucleosyntesis and baryogenesis met in Friedmann cosmology.

(iv) Small particle numbers ($\Omega \leq 1$) of the unwanted massive relics created by the Big Bang and primordial reheating.

(v) Small density of the Friedmann vacuum which is the Λ-term ($\Omega_\Lambda < 0.7$).

First inflationary models were rather connected to the specific physical theories and hypotheses like GUTs, phase transitions, quantum-gravity effects, etc. However, in view of absence of the true high energy physics and,which is more important, taking into account a purely cosmological status of the first three items above, there was an understanding of the necessity in constructing a cosmological standard inflationary theory which could be independent of any current speculations about future fundamental physics, on one side, and could account for the first three puzzles of FU, on the other side. Certainly, such a theory would not solve the fourth an fifth problems which were much more related to the particle physics indeed.

The first theory of such type was proposed by Linde (1983). A basic assumption is that potential energy of inflaton φ-field grows with φ growing (see eq. (92)). The word 'chaotic' minds the requirement for a large value of the initial φ-field ($\varphi \gg 1$) which could be realized somewhere in spacetime under hypothesis of the chaotic initial conditions. However, we do not think that the latter requirement is somehow a problem for the theory at all.

Below, we dwell on the necessary and sufficient conditions for chaotic inflation and then discuss some implications related to the subject.

4.1 Chaotic Inflation

Let us dwell on Lagrangian (66) with the potential term of type (92). To start inflation, the latter must predominate at $\varphi \gg 1$:

$$V(\varphi) > \mid \varphi_{,i}\varphi^{,i} \mid . \tag{129}$$

Let us estimate the size of the region where eq. (129) is initially met.

As we have seen in the previous Chapter the time derivative of initial φ is not a problem regarding the inequality (129) if the spatial homogeneity is postulated, since the inflationary solution is a trap separatrix for $\varphi \gg 1$. So, of principal importance is the spatial gradient condition following from eq, (129):

$$\mid \nabla\varphi \mid \leq H. \tag{130}$$

Eq. (130) can be read as follows: to start inflation one has to prepare a quasi-homogeneous distribution of φ on scale $\sim L = \varphi / \mid \nabla\varphi \mid$ which is much larger than the horizon scale:

$$L \geq l_{SI} = \varphi H^{-1} \gg l_H. \tag{131}$$

The start inflation scale has a physical meaning of the Compton scale of inflaton which becomes explicitly clear in case of the massive field ($V = V_1$):

$$l_{SI} = m^{-1}.$$

Taking it into account for estimates, the start inflation condition (131) is not probably a great surprise in a general case as well.

Eq. (131) deals with the initial distribution of φ-field. In principle, one can rise a question about another (additional to eq. (131)) start inflation condidtion, namely, about initial spatial distribution of metric (curvature) on scales less than l_{SI}. However, we will not discuss this problem here by the formal reason. From the beginning, we decided to restrict ourselves by the case when φ-field is the only source of the metric g_{ik}. The point is that the small-scale nonlinear curvature perturbations (if any) assume another source unrelated to the φ-field since the latter is homogeneous on scale l_{SI}. So, the curvature born by the φ-field is supposed to be homogeneous on scale l_{SI}, as well as the φ-field itself.

Eq. (131) can be interpreted in another way: initial φ-field should be large enough so that l_H could be small. However, more stringent constraints for the potential comes from the slow-roll conditions (103), (105). Let us explicitly rewrite these conditions in terms of the PCP spectrum (106).

From eq. (103) we have the potential restrictions:

$$V^{1/2} \leq 10 q_k. \tag{132}$$

Eqs (105,116) constrict the spectrum index range:

$$\frac{d \ln q_k}{d \ln k} \leq 2. \tag{133}$$

So, q_k cannot vary from the Harrison-Zeldovich spectrum faster than $k^{\pm 2}$ (which is quite compatible with the market of galaxy formation theories considered today).

Eq. (132) puts the direct observational limits on the potential amplitude $V(\varphi)$ for $\varphi = \varphi_k$ within the structure scale range $k^{-1} \sim (10 - 10^4) h^{-1}$ Mpc. Eq. $q_k \sim 10^{-4}$ evidences for the weak coupling of φ-field to the potential $V(\varphi)$ in this region.

Endeed, let us demonstrate it when the c-function variation along the scale range can be negligent. In this case $V = \lambda \varphi^c$, and we obtain a very small value for the coupling parameter (in the Planck units):

$$\lambda \sim 3c^2 10^{-8} \varphi^{-(2+c)} \leq 10^{-8}. \tag{134}$$

Remember that the small coupling parameter is also required for the large size of the FU-bubble (to be more than the horizon today, see below).

Eq. (132) can be used for some other important constraints, e.g., on the reheating temperature. The radiation energy after reheating cannot actually exceed the inflation energy near the end of inflation:

$$\rho_{rad} = \frac{\pi^2}{30} g^* T_{RH}^4 \leq V(\varphi_1) \leq V(\varphi) \leq 10^2 q_k^2, \tag{135}$$

where $\varphi = \varphi_k \geq \varphi_1$ $(k < k_1)$, g^* is the total number of massless degrees of freedom of the thermal bath particles. Eq. (135) gives the following upper limit for the reheating temperature:

$$T_{RH} < \left(\frac{10}{g^*}\right)^{1/4} H^{1/2} \leq \left(\frac{300}{g^*}\right)^{1/4} q_k^{1/2}. \tag{136}$$

Making use the microwave quadrupole anisotropy $q_k \sim \Delta T/T \sim 10^{-5}$, we have for the standard model $(g^* \sim 100)$: $T_{RH} \leq 10^{16}$ GeV. The latter inequality can be confirmed with

help of similar estimate for the gravitational waves produced during inflation (we do not discuss this problem here).

Next important parameter is the Friedmann slice scale l_F, i.e., a typical scale of the part of the Universe, created by inflation, which can be approximated by the Friedmann model. Any scale in such quasi-homogeneous region is described by eq. (117):

$$k = H(\varphi)a(\varphi) = H(\varphi)\exp(-N(\varphi)) \ \ [Mpc^{-1}], \tag{137}$$

where $N(\varphi) = \int Hdt = N_I + N_F$ is the number of e-folds of the Universe expansion from the moment when the perturbation was at the inflationary horizon and up to now

$$N_I = N_I(\varphi) = \int\limits_{\varphi_1}^{\varphi} \frac{\varphi d\varphi}{c}, \qquad N_F \simeq 60. \tag{138}$$

After substituting $\varphi = \varphi_{cr}$ from eq.(128) we have

$$l_F = k_F^{-1} = \exp\left(\frac{\varphi_{cr}^2}{c}\right) \sim \exp\left(\lambda^{-\frac{1}{1+c/2}}\right) \sim \exp(10^4) \gg 10^{28} \ [cm]. \tag{139}$$

The non-linear global Cauchy-Hypersurface which develops in the result of the chaotic inflation dynamics, is not built up yet. Nevertheless, we see no principal difficulties to solve this problem.The point is that the global spatial Cauchy-Hypersurface cannot exist everywhere in spacetime: it breaks in the spacetime regions where the φ-field reaches Planckian densities ($\varphi_{Pl} \sim \lambda^{-1/c}$), so that the semiclassical approach becomes self-inconsistent. We show in the next Section that the latter regions occupy the most part of the physical volume of the Universe produced by the chaotic inflation.

4.2 Stochastic Theory of q-Field

Let us return again to small scales $l_k < l_F$ where q can be treated as a linear quantum operator against the Friedmann background. Equation of motion of q-field is

$$\ddot{q} + 3nH\dot{q} - \frac{1}{a^2}\Delta q = 0. \tag{140}$$

where $n = 1 + 2\dot{\alpha}/3\alpha H$, α and H are the classical background functions, see eqs. (93, 94, 95, 96, 98, 99, 100, 101). As we know from the previous Chapters, the large scale perturbations are classical for $l_k \gg l_H$ while the quantum perturbations affect only small scales, $l_k \leq l_H$. Let us separate these two parts of q-field at the inflation period assuming that q is generated by quantum perturbation:

$$q = \Phi + F, \tag{141}$$

where Φ is the classical large scale part of the q-field operator.

To make this separation explicit let us introduce a notion of the miniuniverse (MU) as a part of the actual space-time of the size proportional to the horizon:

$$l_{MU} = \zeta H^{-1} \geq l_H, \tag{142}$$

where $\zeta = \text{const} \geq 1$. Evidently, MUs do not expand with the comoving volume.

Now, we can define the classical Φ-field as the mean value of q in MU:

$$\Phi = \Phi(t, \vec{x}) = \int K_\sigma(\vec{x} - \vec{x}^{\prime}) q(t, \vec{x}^{\prime}) d^3\vec{x}^{\prime}, \tag{143}$$

where $K_\sigma(\vec{r}) = (2\pi)^{-3/2}\sigma^{-3}\exp(-r^2/2\sigma^2)$ is the Gaussian MU-window, and

$$\sigma = \frac{l_{MU}}{a} = \frac{\zeta}{aH} = -n_1\zeta\eta$$

is the MU-dimension in the comoving $\vec{x}$-space, $n_1 = (-a\eta H)^{-1}$.

Similarly to Φ we can define the classical part of the q-field momentum:

$$V = V(t, \vec{x}) = \int K_\sigma(\vec{x} - \vec{x}^{\prime}) \dot{q}(t, \vec{x}^{\prime}) d^3\vec{x}. \tag{144}$$

The evolution of the coarse grained fields Φ and V is governed by the quantum perturbations presented by the F-operator: in the inflationary process new and new perturbations created inside the horizon inflate, one followed by another, outside the horizon and start contributing to the classical fields Φ and V when their scales become about (and then larger) than l_{MU}. So, F plays a role of the stochastic generator for Φ, the latter moving like a Brownian particle in the gas. More of this, the dynamical equations are similar as well.

Let us present eqs. (143,144) as the Fourier integrals (see eq. (61)):

$$\Phi = \int d^3\vec{k}\ \ \Theta(a_{\vec{k}} q_{\vec{k}} + a^\dagger_{\vec{k}} q^*_{\vec{k}}),$$

$$V = \int d^3\vec{k}\ \ \Theta(a_{\vec{k}} \dot{q}_{\vec{k}} + a^\dagger_{\vec{k}} \dot{q}^*_{\vec{k}}), \tag{145}$$

where

$$\Theta = \Theta(k\sigma) = e^{-\frac{1}{2}k^2\sigma^2} = \int K_\sigma(\vec{r}) e^{i\vec{k}\vec{r}} d^3\vec{r}.$$

Then the original eq. (140) can be rewritten in terms of the classical fields:

$$\dot{\Phi} - V = f,$$

$$\dot{V} + 3nHV = \frac{Hg}{\zeta^2}, \tag{146}$$

f and g can be called the noise functions (or generators) driven by the quantum fluctuations:

$$f = mH\sigma^2 \int d^3\vec{k}\ \ k^2\Theta(a_{\vec{k}} q_{\vec{k}} + a^\dagger_{\vec{k}} q^*_{\vec{k}}),$$

$$g = H\sigma^2 \int d^3\vec{k}\ \ k^2\Theta(a_{\vec{k}} p_{\vec{k}} + a^\dagger_{\vec{k}} p^*_{\vec{k}}), \tag{147}$$

where

$$p_{\vec{k}} = -q_{\vec{k}} + \frac{m\zeta^2}{H}\dot{q}_{\vec{k}}, \quad m = -\frac{\dot{\sigma}}{\sigma H}.$$

Before we calculate correlators of the noise functions let us introduce the normal modes of the classical fields,

$$\tilde{\Phi} = \Phi + \frac{m_1}{3H}V, \quad \tilde{V} = V,$$

and separate the Eqs. (146):

$$\dot{\tilde{\Phi}} = \tilde{f}. \tag{148}$$

(Eq. (146) for V-function did not change). Here $\tilde{f} = f + m_1 g/\zeta^2$. The background functions $n_{(1)}$ and $m_{(1)}$ are easily derived on the inflationary separatrix ($\varphi \gg 1$):

$$\begin{aligned} n &= 1 + c(1-e)\varphi^{-2} + O(\varphi^{-4}), \\ n_1 &= 1 - \frac{1}{2}c^2\varphi^{-2} + O(\varphi^{-4}), \\ m &= 1 - \frac{1}{2}c^2\varphi^{-2} + O(\varphi^{-4}), \\ m_1 &= 1 - \frac{1}{6}c^2\varphi^{-2} + O(\varphi^{-4}). \end{aligned} \tag{149}$$

Eq. (148) coincides with the Langeven equation describing the drift of a Brownian particle if the particle coordinate is understood instead of Φ.

When comparing windows Θ and $k^2\Theta$ for the classical and noise functions, we can see that in the latter case the main contribution comes from scales $l \sim l_{MU}$. It is clear: the field averaged over the mini-universe, can change its value not before the new perturbation reaches the MU-scale which happens in a characteristic time (step-time) $\Delta t \sim l_{MU}$. Since the perturbations phases appearing on the MU-scale are random, the process of the classical field change is stochastic. To calculate the characteristic values of this process we must know the correlators of the noise functions on the Friedmann hypersurface $t = \text{const}$.

If the q-field is in the vacuum state then

$$\langle a^{\dagger}_{\vec{k}} a_{\vec{k}'} \rangle = 0, \quad \langle a_{\vec{k}} a^{\dagger}_{\vec{k}'} \rangle = \delta(\vec{k} - \vec{k}'),$$

and the process is the Gaussian one, so that, the second correlators are quite sufficient to know about. In this case the perturbation amplitudes for $\varphi > 1$ and $\zeta > 1$ are as follows (see eqs. (61,115,118)):

$$q_{\vec{k}} = i\sqrt{2}\left(\frac{H\varphi}{c}\right)_k \exp(i\vec{k}\vec{x}), \quad \frac{\zeta^2}{H}\dot{q}_{\vec{k}} = k^2\sigma^2 q_{\vec{k}}.$$

Finally, after the straightforward calculation we have in the main approximation over φ and ζ:

$$\langle \tilde{f}\tilde{f}' \rangle = \langle ff' \rangle = \frac{1}{2}(\langle fg' \rangle + \langle gf' \rangle) = \langle gg' \rangle = 2D\delta(t - t'), \tag{150}$$

where $D = D(t) = H^3(\varphi/2\pi c)^2$ is the diffusion coefficient. The δ-function in eq. (150) is used instead of each of the following expressions,

$$2H\left(\frac{\sigma}{\sigma'} + \frac{\sigma'}{\sigma}\right)^{-2} \quad \text{and} \quad 12H\left(\frac{\sigma}{\sigma'} + \frac{\sigma'}{\sigma}\right)^{-4},$$

because the halfwidths of the latter bell-functions are about the cosmological horizon ($\Delta t \sim H^{-1}$) which is less (by the ζ-factor) than the MU-scale.

Now, we can introduce the probability distribution $P = P(t, \tilde{\Phi})$ to find field $\tilde{\Phi}$ at time t. By definition,

$$\int P d\tilde{\Phi} = 1.$$

Following the standard methods, we can derive the Fokker-Planck equation for this function:

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial \tilde{\Phi}^2}. \tag{151}$$

Obviously, the field dispersion grows in time in this stochastic process.

Let us take some arbitrary MU at time t_o with the classical field $\tilde{\Phi}_o$, which we call the mother. During inflation the physical spatial volume which belonged to the mother MU at $t \sim t_o$, expands to larger and larger scales. For $t > t_o$, this volume can be covered by other MUs (daughters) [15]. The $\tilde{\Phi}$-field varies from one daughter to another, and the r.m.s. deviation (from $\tilde{\Phi}_o$) $\tilde{\sigma} = \tilde{\sigma}(t)$ can be calculated making average either by the quantum q-state in one MU or over the daughter MUs assembly:

$$\tilde{\sigma}^2 = \langle(\tilde{\Phi} - \tilde{\Phi}_o)^2\rangle = \int (\tilde{\Phi} - \tilde{\Phi}_o)^2 P d\tilde{\Phi} =$$

$$= 2\int D dt = -\frac{1}{2\pi^2}\int \frac{V^4 d\varphi}{(dV/d\varphi)^3} \sim t. \tag{152}$$

It can be confirmed also by the exact solution of eq. (151)

$$P(t, \tilde{\Phi}) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp\left(-\frac{(\tilde{\Phi} - \tilde{\Phi}_o)^2}{2\tilde{\sigma}^2}\right). \tag{153}$$

At the beginning

$$\tilde{\sigma}(t_o) = 0, \quad P(t_o, \tilde{\Phi}) = \delta(\tilde{\Phi} - \tilde{\Phi}_o),$$

while, during time, the distribution (153) broadened around $\tilde{\Phi}_o$ with equal probability for both signs of the deviation $(\tilde{\Phi} - \tilde{\Phi}_o)$. The typical one-step-change of the $\tilde{\Phi}$-field is

$$\Delta(\tilde{\Phi}) = \tilde{\sigma}(\Delta t) = \sqrt{\frac{\zeta D}{H}} \sim \sqrt{\zeta} H. \tag{154}$$

Before we discuss some implications of this stochastic process, let us consider the necessary conditions for the diffusion approach.

Eqs. (148,150) are generally true if the D-function varies slower than the characteristic step-time $\Delta t \sim \zeta H^{-1}$. This requirement is commonly satisfied on the inflationary separatrix.

The Fokker-Planck approach (151) is less reliable here. Indeed, the notion of the P-function assumes that its characteristic change-time should be no larger than the step-time. However, regarding eq. (153), it can be marginally so if ζ is not too high. Below, we will assume that $\zeta \sim 1$.

4.3 Non-Linear Inflation

Mini-universes of size $H^{-1}(\zeta \sim 1)$ introduced in Section (4.2), are just the causally connected regions of the inflating space. If any two points with constant comoving $\vec{x}$-coordinates (i.e., expanding with the Universe) belong initially to the same MU then they will manage to exchange the light signals at least once. But if they belong to two different MUs then the light

[15] At each step $\Delta t \sim \zeta H^{-1}$ the mother volume expands by factor $N \sim \exp(3\zeta)$, so there are about N daughters of the first generation, N^2 of the second, and so on so forth, inside the volume.

signal sent from one point will never get the other one. An important consequence is that each MU expands in time independently of any other.

It means that any MU can be chosen as mother regarding the next generations of the daughter MUs it produces. In its turn, any daughter taken at a moment $t \sim t_o$, although created by some mother at $t < t_o$, is the mother itself for $t > t_o$. This picture has neither beginning no end. Actually, this boiler of MUs is eternally self-reproducing inflationary Universe.

Let us consider some physical volume expanding with the comoving space. New and new MUs are created inside the volume during the evolution. We can connect by the time-like tracks causally related MUs (mother-daughter, mother-daughter, etc.). We saw in the previous Section that there is an equal probability for both signs of the field deviation $\Delta\tilde{\Phi}$ to be found along any track from the past to future. Since any MU develops independently of the previous history and its neighbours, we can forget about the seed mother field $\varphi^{(o)}$ and try to find the current classical quasi-Friedmannian field φ_{MU} driven the given MU on the track and applicable only to this MU. Certainly, φ_{MU} is the local φ-field renormalized each time by the classical part of q-field (see eq. (141)).

Technically, we can use Newtonian gauge to find φ_{MU}, since this particular frame most closely imitates the local Friedmannian expansion (see eqs. (53)):

$$ds^2_{MU} = \left(1 - \frac{H}{a}P_{MU}\right)dt^2 - a^2\left(1 + \frac{H}{a}P_{MU}\right)dx^2, \tag{155}$$

where

$$P_{MU} = \int a\gamma\Phi dt.$$

The local time is

$$t_{MU} = t - \int \frac{H}{2a}P_{MU}dt.$$

and the local expansion factor is

$$a_{MU} = a\left(1 + \frac{H}{2a}P_{MU}\right).$$

In both equations we disregarded the dependence of the P_{MU} function on the $\vec{x}$-coordinates within MU. In doing so, we can easily recover the φ_{MU}-field from background eqs. (95). In the main approximation over $\varphi \gg 1$, we have:

$$\frac{\varphi_{MU}}{\varphi^{(o)}} = 1 + \frac{1}{c}\left(\gamma\Phi - \frac{H}{a}P_{MU}\right). \tag{156}$$

Below, we give only qualitative ideas about some results of this investigation.

The most interesting question arisen is as to which densities of the φ-field most tracks lead during stochastic inflation? The answer is a production of two factors: the probability to find a certain φ-field on one track, and the number of tracks carring given φ-field. As far as the first factor is concerned, the classical monotonic decrease of φ when sliding down the potential $V(\varphi)$, is the smaller the larger the field is: $\Delta\varphi_{cl} \sim -c/\varphi$ for one step-time $\Delta t \sim H^{-1}$. On the other hand, the field stochastic change due to the quantum perturbations proceeds in both directions of φ with amplitude $\mid \Delta\varphi_{st} \mid \sim H$ growing to higher field values. So, for large enough $\varphi(\varphi > (c^2/\lambda)^{1/(2+c)})$ the quantum stochastic process predominates, so that, at each step half of the created daughters have higher φ than their mother. But the total number of tracks created per unit time grows to higher fields as well, $dN/dt \simeq 20H$. So, the majority of tracks leads

to high densities, thus, the largest part of global physical volume is occupied by the Planckian field density, i.e., by the space-time foam.

We can only guess what is happenning there, in these most typical high density states of the global Universe, — the notions of the space-time and inflaton break, mutable transitions to different physics, signatures, dimensions and other conceivable and inconceivable worlds may occur. In fact, we can only say that the inflation states with densities less than the Planckian one, are non-typical and very unprobable ones in this really chaotic Universe dominated by the sea of quantum fluctuations. The stochastic regime considered above is just a part, the semiclassical part of this sea, where the space-time is already classical while the inflaton is still dominated by quantum fluctuations. Evidently, such *regions* decouple occasionally from the space-time foam and exist independently during some period of the *classical time*.

A very important conclusion is that inside these semi-classical regions there exist some very few tracks which lead occasionally (through the random stochastic process) to lower and lower densities of φ-field. When the latter becomes below the critical quantity ($\varphi \leq (c^2/\lambda)^{1/(2+c)}$), quantum fluctuations are not able any more to increase φ in the created daughter- MUs and the successive inflation continues with a monotonic decrease of the inflaton. Now, the quantum fluctuations are responsible only for small density perturbations varing slightly from one daughter to other. So, the result will be the adiabatic Gaussian perturbations (with the amplitude decreasing to smaller scales) against the Friedmannian background patch surrounded by the non-linear chaotic Universe.

Let us emphasize two other points in the conclusion.

The global Cauchy-Hypersurface does not rigorously exist. It can be constructed near the Friedmannian patch in a space-time region restricted by the Planckian densities. In fact, such a non-linear solution describes just a temporal island formed with a very small probability in the chaotic space-time foam of the Universe and suitable for life. In this connection, we can mention that we do not think it is worth while putting seriously the question as to how general are the start-inflation conditions for the chaotic or other inflation theories from the point of view of general solutions of the CGR equations? In our opinion, it is quite enough that the probability for creating a low-density world where life can appear, is non-zero. We have no time to discuss this subject in more detail here.

The next important point is as follows. There are the inflaton quantum fluctuations in the chaotic inflation theory that are the reason for both, the non-linear global structure of the Universe as a whole and the adiabatic density perturbations responsible for the large scale structure formation in the Friedmannian patch of such a Universe. Therefore, we can test the inflationary theories just investigating the spectrum of PCPs and then reconstructing the global structure within the theory frameworks. This is, probably, the only informative cosmological channel to learn anything about the features beyond optical horizon, as well as the fundamental physics parameters beyond direct experiment. We can see that the problem of testing inflation becomes part and parcel of the VEU theories. We are going to discuss it briefly in the next Chapter.

5 Testing VEU

There are few important cosmological predictions coming from the very early inflationary epoch. Among them are the total energy density in the Universe and the spectrum of adiabatic PCPs.

The former quantity is to be equal to critical density up to accuracy of the PCP amplitude

on the contemporary horizon:

$$\Omega_{tot} = 1 \pm \delta_H, \quad | K | \leq \delta_H. \tag{157}$$

This amplitude $\delta_H = \delta_k(k = H_o)$ can be easily estimated by the quadrupole anisotropy of the microwave background radiation:

$$\delta_H \sim 10^{-4}.$$

The real dynamical density is close to the critical one with the accuracy $\sim 30\%$. However, even if eq. (157) were confirmed with a much higher accurary by future observations, it could not, unfortunately, tell us much information about the inflation principal parameters. In this respect, more informative is the PCP spectrum generated by last stages of the inflationary epoch before the beginning of FU.

As far as the chaotic inflation is concerned, the postinflationary PCP spectrum is very sensitive to the potential form (see eqs. (116, 117). On the other hand, the shape of the potential energy of the inflaton found in a given Particle Physics, must be unambiguously fixeds by the fundamental Lagrangian regarding all the particle fields and interactions.

Thus, a principle test of inflation is the large scale structure of the Universe. Its analysis allows for restoring of the postrecombination PCP spectrum on galactic to horizon scales. This part of the spectrum is obviously related to the postinflationary PCPs which, in their turn, depend directly on the inflaton potential within the field interval responsible for the scales mentioned above (see eq. (117)).

Two following questions arise from this consideration:

* * How to relate the postinflationary and postrecombination spectra?
* ** Which are the basic cosmological observations now pouring the light on the PCP spectrum?

Below, both topics are discussed very briefly.

5.1 Transfer Functions

The point is that any real confrontation of theory with observations can be done only within the framework of some cosmological model allowing to transfer PCPs from times when they were produced and up to the moment when they entered the non-linear evolution to form the hierarchy of the objects observed. The principal parameters of the model are those of the dark matter components running out the gravitational evolution of PCPs.

Regarding the gravitational impact to PCPs there are two components of the Dark Matter Cold (CDM) and Hot (HDM) that are important. The cold particles may be heavy relics ($m_C \gg 10eV$) or coherent axions which behave like a non-relativistic medium. The hot particles are those like massive neutrino with the equilibrium particle density and according restmass ($m_H \leq 10\ eV$). These two components evolving very differently in the past are both non-relativistic and maintain the critical density now:

$$\Omega_C + \Omega_H + \Omega_\Lambda = \Omega_{tot} = 1. \tag{158}$$

Baryons can be added in Ω_C, the third term which is the energy of physical vacuum cannot be totally excluded today $0 < \Omega_\Lambda < 0.7$.

The rest dark matter, which we call as the ν-component, does not contribute crucially to eq.(158) and consists of relativistic and semirelativistic weakly interacting particles ($m_\nu \ll 1eV$). We can characterize ν-particles by there total number N_ν with respect to the relic photons:

$$\nu = \frac{N_\nu}{N_\nu + N_\gamma} \in (0,1). \tag{159}$$

For the standard CDM and HDM models $\nu = 0.4$ and $\nu = 0.3$ respectively, counting three or two sorts of the massless neutrino. Generally, ν-particles include gravitions, light SUSY and other hypothetical ...inos probably existing in the Universe.

So, in the simplest matter dominated case ($\Omega_{tot} = 1,\ \Lambda = 0$, stable particles) we have two free parameters, Ω_H and ν, both ranging from zero to one, which determine the past history of PCPs beginning from Inflation. The goal is to find the ratio of the final to initial PCP spectra as a function of these two (or more in a general case) parameters. This ratio is called the transfer function,

$$T(k) = \frac{q_k^{(f)}}{q_k^{(i)}}, \tag{160}$$

where postinflationary spectrum $q_k^{(i)}$ coincides with the function q_k from eqs. (116, 117), $q_k^{(f)}$ is the postrecombination PCP spectrum responsible for LSS formation in the Universe. Evidently, $T(k)$ does not depend on the inflationary period, it is a functional of the Friedmann model dynamics from the beginning to our days.

$T(k)$ is equal to unity for very large scales ($T(0) = 1$) and then decreases monotonically with k growing. It still remains to be about unity up to some characteristic scale l_{eq} which coincides with the horizon at period at equality of all relativistic and all non-relativistic component densities. The further $T(k)$-fall-shape to shorter wavelengths is an intrinsic property of the model (we refer this subject to the special courses and lectures).

For the standard models $l_{eq} \sim 30h^{-2}$ Mpc but, for arbitrary ν, l_{eq} grows with ν growing like $l_{eq} \sim (1-\nu)^{-1/2}$.

As we see, the resulting spectrum $q_k^{(f)}$ is a sensitive function of both fundamentals of the early Universe, the postinflation PCPs and dark matter composition. So, the investigation of its direct creature — the large scale structure of the Universe — cannot be overestimated today. Another principle test — a laboratory detection of the dark matter particles — is not discussed here.

5.2 Observations

Since there are few special courcses devoted to this subject we only briefly outline the hot spots of the confrontation between observations and theory important for us. We see tremendous importance for the modern cosmology of two groups of experiment nowdays: $\Delta T/T$ observations both on large and small scales ($\Theta \lesssim 1^o$) and direct observations of the distribution and evolution of the hierarchies of LSS.

The point is that both experiments confront and complement each other.

If $\Delta T/T$ upper limits and detections which become available now, make us to lower down the primordial perturbation amplitudes on scales larger than $l > 10h^{-1}$ Mpc, then LSS needs for its existence high enough cosmological perturbation amplitudes on scales $l \sim 10 - 100h^{-1}$ Mpc. For the most theories of galaxy formation the gap between these two requirements is quite negligible. Say, within the Gaussian perturbation theories any reasonable asumption for

large superclusters and voids to be more or less standard phenomenon in the visible Universe, leads inevitably to the $\Delta T/T$ prediction levels on degrees of arc capable for current detection. It brings a very great optimism to obtain large scale primordial perturbation spectrum directly from the observations with a high degree of accuracy.

The current situation with $\Delta T/T$ is well known. For our case of Gaussian PCPs we may directly relate our scalar q on the last scattering surface at recombination with the map of the temperature anisotropies on the selestial sphere $\vec{n} = \vec{n}(\theta, \varphi)$. Endeed, let us decompose the latter in spherical functions:

$$\frac{\Delta T}{T}(\vec{n}) = \sum_{l,m} a_{lm} Y_{lm}(\theta, \varphi). \tag{161}$$

Then, after simple calculations, the temperature correlation function takes the following form:

$$C(\alpha) \equiv \langle \frac{\Delta T}{T}(\vec{n}_1) \frac{\Delta T}{T}(\vec{n}_2) \rangle = \sum_l C_l P_l(\cos\alpha), \tag{162}$$

where $\vec{n}_1 \vec{n}_2 = \cos\alpha$, $P_l(\cos\alpha)$ are the Legandre polinomials, $\langle ... \rangle$ is the average over the field state, and

$$C_l = \frac{2l+1}{4\pi} a_l^2 = \frac{1}{4\pi} \sum_m \langle a_{lm}^2 \rangle. \tag{163}$$

Generally, there are three main sources of the primordial temperature anisotropies: due to fluctuations of the gravitational potential, matter density and velocity perturbations. For the vanishing pressure they may be reduced to the following expression taken at recombination:

$$\frac{\Delta T}{T}(\vec{n}) = \frac{1}{3} q(\vec{x}_r) + \frac{1}{4} \delta_{\gamma,r} - \vec{n}\vec{v}_r. \tag{164}$$

The first term (Sachs-Wolf effect) dominates for $\alpha > 1^o (l > 100\ h^{-1}$ Mpc) which yields the direct connection between $C(\alpha)$ and the PCP spectrum q_k:

$$a_l^2 = \frac{2}{\pi} \int \frac{dk}{k} q_k^2 J_l^2(k), \quad C(\alpha) = \frac{1}{2\pi^2} \int \frac{dk}{k} q_k^2 J_o(k\beta), \tag{165}$$

where J_l are the Bessel functions, $\beta = 2\sin\frac{\alpha}{2}$.

The most important is the COBE detection for $\alpha \sim 10^o$ evidencing the consistency with the HZ-spectrum on very large scales, $l \sim 1000\ h^{-1}$ Mpc:

$$\Delta_k^2 \sim k^4 q_k^2 \sim k^\alpha, \quad \alpha = 4 \pm 0.2, \tag{166}$$

where $\delta^2 = (1+z)^{-2} \int \frac{dk}{k} \Delta_k^2$ is the mean square perturbation of density.

As for the LSS data, we have many independent indications today for existence of the large scale structures up to a typical scale $l_{LS} \simeq 100\, h^{-1} Mpc$. The most important data come from the Great Attractor ($z \leq 0.03$), pencil beam galactic surveys ($z \leq 0.3$), existence of large groups of quasars ($z \in (0.5, 2)$), and spatial distributions of clusters of galaxies ($z \leq 0.1$). These LSS experiments indicate the following estimate for the spectrum of Gaussian density perturbations within the scales $l \in (10, 100)\, h^{-1}\, Mpc$:

$$\Delta_k^2 \sim k^\gamma, \quad \gamma = 0.9 \pm 0.2. \tag{167}$$

The consistency with eq.(166) gives us the obvious evidence for the presence of a real feature in the power spectrum at the supercluster scale $\sim 100\, h^{-1}\, Mpc$; it may a change in the spectrum

slope from HZ (at $l > 100\,h^{-1}\,Mpc$) to the flat one ($l > 100\,h\,Mpc$). This "signature of the God" requires its explanation in physics of the very early Universe.

We do not discuss here modern status of the cosmological model and observational tests refering to special reviews (e.g. astro-ph/9803212).

6 Conclusion

As we have seen, still in the absence of high energy physics, we may successfully develop the theory of VEU on purely cosmological grounds and come to important conclusions about the spectrum of PCPs capable of current testing by observations. On the other hand, we are very close to recover the postrecombination PCP spectrum directly from the observations, both $\Delta T/T$ and LSS (especially its evolutionary aspects), and thus to reconstruct the true cosmological model and make the exciting link to VEU physics. Both confronting branches – theory and observations – develop fruitfully and make us hope to find the principal answers on the evolutionary model of the Universe during our days.

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Appendix A

Here we obtain the Lagrangian $L^{(2)}$ (see eqs. (26)).

Asuming that eqs. (25) are exact ones, all the auxiliary quantities are expanded to the second order in ϕ and h^{ik}:

$$\delta w = \frac{1}{2w^{(o)}}(\delta(\varphi_{,i}\varphi_{,k}g^{ik}) - (\delta w)^2) = w^{(o)}(\chi - \frac{1}{2}\chi^2 + \frac{1}{2}v_i v^i - u_i v_k h^{ik}),$$

$$\delta L = n^{(o)}(\delta w - \nu^{(o)}\phi + \Gamma\frac{\delta w}{w}\phi + \frac{1}{2w}((\frac{\delta w}{\beta})^2 - m^2\phi^2) =$$

$$= n^{(o)}w^{(o)}(\chi - \nu^{(o)}v + \Gamma\chi v + \frac{\chi^2}{2}(\beta^{-2} - 1) + \frac{1}{2}v_i v^i - u_i v_k h^{ik} - \frac{1}{2}m^2 v^2)$$

$$\delta g_{ik} = h_{ik} + h_{il}h^l_k, \quad \ln\left(\frac{g}{g^{(o)}}\right) = h + \frac{1}{2}h^k_l h^l_k,$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g^{(o)}}\;(h + \frac{1}{2}h^k_l h^l_k + \frac{1}{4}h^2),$$

$$\delta\Gamma^l_{ik} = \frac{1}{2}(h^l_{i;k} + h^l_{k;i} - h^{;l}_{ik} + h^m_i h^l_{m;k} + h^m_k h^l_{m;i} + h^l_m h^{;m}_{ik} - (h_{im}h^m_k)^{;l}),$$

$$\delta\Gamma^l_{il} = \frac{1}{2}(h + \frac{1}{2}h^k_l h^l_k)_{,i},$$

$$\delta R = g^{ik}\delta R_{ik} - h^{ik}R^{(o)}_{ik},$$

$$\delta R_{ik} = (\delta\Gamma^l_{ik})_{;l} - (\delta\Gamma^l_{il})_{,k} + (\delta\Gamma^l_{ik})(\delta\Gamma^m_{lm} - (\delta\Gamma^m_{il})(\delta\Gamma^l_{km}),$$

where $\delta f = f - f^{(o)}$, $v = \phi/w^{(o)}$, $v_i = \phi_{,i}/w^{(o)}$, Γ^l_{ik} are the Christoffel symbols.

The substitution to eq. (20) yields:

$$(L - \frac{1}{2}R)\sqrt{\frac{g}{g^{(o)}}} = L^{(o)} - \frac{1}{2}R^{(o)} + \frac{1}{2}(1 + \frac{1}{2}h)(G^{(o)}_{ik} -$$

$$-T^{(o)}_{ik} - \phi((n^{(o)}u^l)_{;l} + n^{(o)}\nu^{(o)}) + S^l_{;l} + L^{(2)},$$

$$S^l = \frac{1}{2}(1+\frac{1}{2}h)(g^{il}\delta\Gamma^k_{ik} - g^{ik}\delta\Gamma^l_{ik}) + n^{(o)}u^l\phi,$$

$$L^{(2)} = \frac{1}{2}nw(v_iv^i + \chi^2(\beta^{-2}-1) - 2v_i\psi^i_ku^k + \nu v\psi -$$

$$-m^2v^2 + 2\Gamma v\chi) + \frac{1}{4}(L^o - \frac{1}{2}R^{(o)})(\psi_{ik}\psi^{ik} - \frac{1}{2}\psi^2) +$$

$$+\frac{1}{8}(\psi_{ik;l}\psi^{ik;l} - 2\psi_{ik;l}\psi^{il;k} - \frac{1}{2}\psi_{,l}\psi^{,l}).$$

If background metric satisfies the classical equations,

$$G^{(o)}_{ik} - T^{(o)}_{ik} = 0, \quad (n^{(o)}u^l)_{;l} + n^{(o)}\nu^{(o)} = 0,$$

then the substitution to $L^{(2)}$ gives eq. (26). Note, that the linear terms in ϕ and h^{ik} (see eq. (20)) prove to be zero since the background equations are met.

Appendix B

Here we derive general relations between q-scalar and the perturbations in arbitrary reference frame.

Let us decompose perturbations (25) over the irreducible representations of potential type in general Friedmann model with the metric g_{ik} and 4-velocity u^i:

$$v = X + \frac{1}{2}(C + D_{,i}u^i),$$

$$h_{ik} = Ye_{ik} + Zg_{ik} + (Cu_{(i)})_{;k)} + D_{;ik}, \tag{169}$$

where 4-tensor $e_{ik} = 2u_iu_k - g_{ik}$ has Euclidean signature and subbrackets mean the symmetrization. Functions X, Y, Z, C, D are coefficients of the linear decomposition. X, Y and Z are gauge invariant 4-scalars. C and D are arbitrary functions specifying the gauge feedom of perturbations in eqs. (25) (see eqs. (28, 29)):

$$\xi_i = \frac{1}{2}(Cu_i + D_{,i}).$$

The Einstein linear equations can be decomposed as a 4-tensor over the irreducible representations as well:

$$h^{;l}_{ik;l} - 2h^l_{(i;k)l} + h_{;ik} + (p-\epsilon)h_{ik} +$$

$$+(\epsilon+p)(\tilde{\chi}(\beta^{-2}-1)e_{ik} + 4v_{(i}u_{k)} - 2\nu vg_{ik}) =$$

$$= Ee_{ik} + Fg_{ik} + (Iu_{(i)})_{;k} + J_{;ik} = 0,$$

where

$$E = Y^{;l}_{;l} - 4HY_{,l}u^l + (p - \epsilon + 4(\dot{H} + H^2))Y +$$

$$+(\epsilon+p)(\tilde{\chi}(\beta^{-2}-1) + 2(4H+\nu)X),$$

$$F = Z^{;l}_{;l} + (p-\epsilon)Z - 4H(Y_{,l}u^l + HY - (\epsilon+p)X),$$

$$I = -2(Y_{,l}u^l + HY - (\epsilon + p)X),$$
$$J = 2Z,$$

and $\tilde{\chi} = w^{-1}(wX)_{,l}u^l - (Y + Z)/2$. Here, the auxiliary relations

$$u_{i;k} = HP_{ik} = \frac{1}{2}H(g_{ik} - e_{ik}), \quad H = \frac{1}{3}u^l_{,l},$$

and some other background formulae were used.

From $J = 0$ we have [16]:

$$Z = 0 \tag{170}$$

From $I = F = 0$ the following relation between X and Y scalars is obtained:

$$\dot{Y} + HY = (\epsilon + p)X. \tag{171}$$

For the spatially flat model ($K = 0$) q-scalar is given by the linear superpositions of the gauge invariant functions:

$$q = -(Y + 2HX) \tag{172}$$

Obviously, it is the real 4-scalar (as well as X and Y) independent of any reference frame.

The inverse transformations follow from eqs. (171, 172):

$$X = \frac{1}{2}\left(\frac{P}{a} - \frac{q}{H}\right), \quad Y = -\frac{H}{a}P, \tag{173}$$

where $P = \int a\gamma q dt$, $\gamma = -\dot{H}/H^2$.

From $E = 0$ we have the key equation [17]:

$$a^3\gamma\beta^{-2}\dot{q} = \Delta P. \tag{174}$$

The rest is to prescribe the potentials C and D for different gauges.

Projecting eqs. (169) on the Friedmann reference system we have for an arbitray gauge:

$$v = X + \frac{1}{2}F, \quad h_{oo} = Y + \dot{F},$$
$$h_{o\alpha} = \frac{1}{2}\psi_{,\alpha}, \quad h^\beta_\alpha = A\delta^\beta_\alpha + B^{;\beta}_{,\alpha}, \tag{175}$$

where $F = C + \dot{D}$, $\psi = a^2(D/a^2)^{\dot{}} + F$, $A = HF - Y$, $B = -D/a^2$. So, the most general definition of q-scalar in terms of 3-potentials is the following (cf. eq. (34)):

$$q = A - 2Hv. \tag{176}$$

For the orthogonal gauge ($\psi = 0$):

$$C = 2a(aB)^{\dot{}}, \quad D = -a^2 B, \quad F = a^2\dot{B}. \tag{177}$$

The next examples specify the function B in eqs. (177).

For the synchronous gauge ($h_{oo} = \psi = 0$):

$$\dot{B} = a^{-2}Q - a^{-3}P, \quad Q = \int \gamma q dt. \tag{178a}$$

For the comoving gauge ($v = \psi = 0$):

$$\dot{B} = \frac{q}{Ha^2} - a^{-3}P. \tag{178b}$$

For the Newtonian gauge $B = \psi = 0$ (cf. eqs.(51, 52, 53)).

[16]The scalar J is generally connected to the anisotropic pressure which is zero for φ-field

[17]Functions $(\epsilon + p)\tilde{\chi}$ and $(\epsilon + p)X$ should be excluded from E with help of eq. (171).

Appendix C

One can easily verify that eqs. (42) are just the linear expansion terms of the exact solution

$$ds^2 = dt^2 - a^2 \exp(2\tau a_{\alpha\beta}) dx^\alpha dx^\beta, \quad \tau = \int \frac{dt}{a^3}, \tag{179}$$

where the function $a = a(t)$ can be found as follows:

$$H^2 = \frac{1}{3}\epsilon - \frac{1}{9a^6}\Lambda^2, \quad \Lambda^2 = \frac{2}{3} a^\beta_\alpha a^\alpha_\beta,$$

$$\dot{H} = -\frac{1}{2}(\epsilon + p) + \frac{1}{3}\frac{\Lambda^2}{a^6}.$$

For $t \to 0$, we have the Kazner asymptotic:

$$a^3 = \Lambda t, \quad g_{\alpha\beta} \sim \mathrm{diag}(t^{2p_a}), \tag{180}$$

where Kazner exponents ($p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$) are obviously related to the eigen values of matrix $a_{\alpha\beta}$:

$$det(a_{\alpha\beta} - \lambda\delta_{\alpha\beta}) = 0, \quad \lambda_a = \Lambda\left(p_a - \frac{1}{3}\right).$$

Eqs. (170) describe solution for the Bianchi type I model with comoving space. The infinite scale vortex and gravitational-wave perturbations lead also to eqs. (179). Note, that the infinite scale perturbations although causing the expansion anisotropy (shear), do not perturb the spatial curvature and density perturbations ($\delta\epsilon = u_\alpha = 0$). It happens only in spatially flat Friedmann models.

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